MAT 441 **Problem Set #5** Name SOLUTIONS

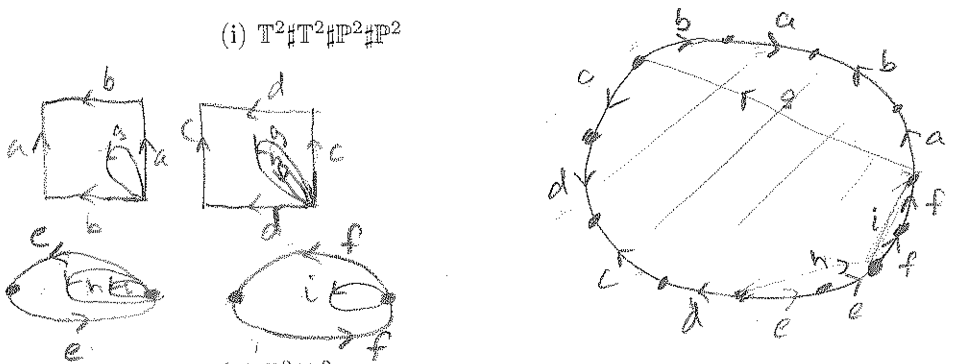
4/6/09 (due Monday 4/13/09)

15 points

Justify all answers. Unsupported answers may not receive full credit.

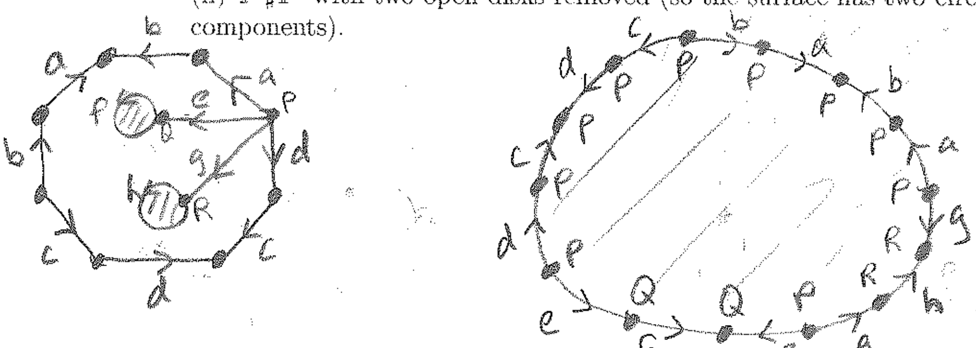
1. (a) Find a planar identification diagram for

(i) $\mathbb{T}^2\sharp\mathbb{T}^2\sharp\mathbb{P}^2\sharp\mathbb{P}^2$



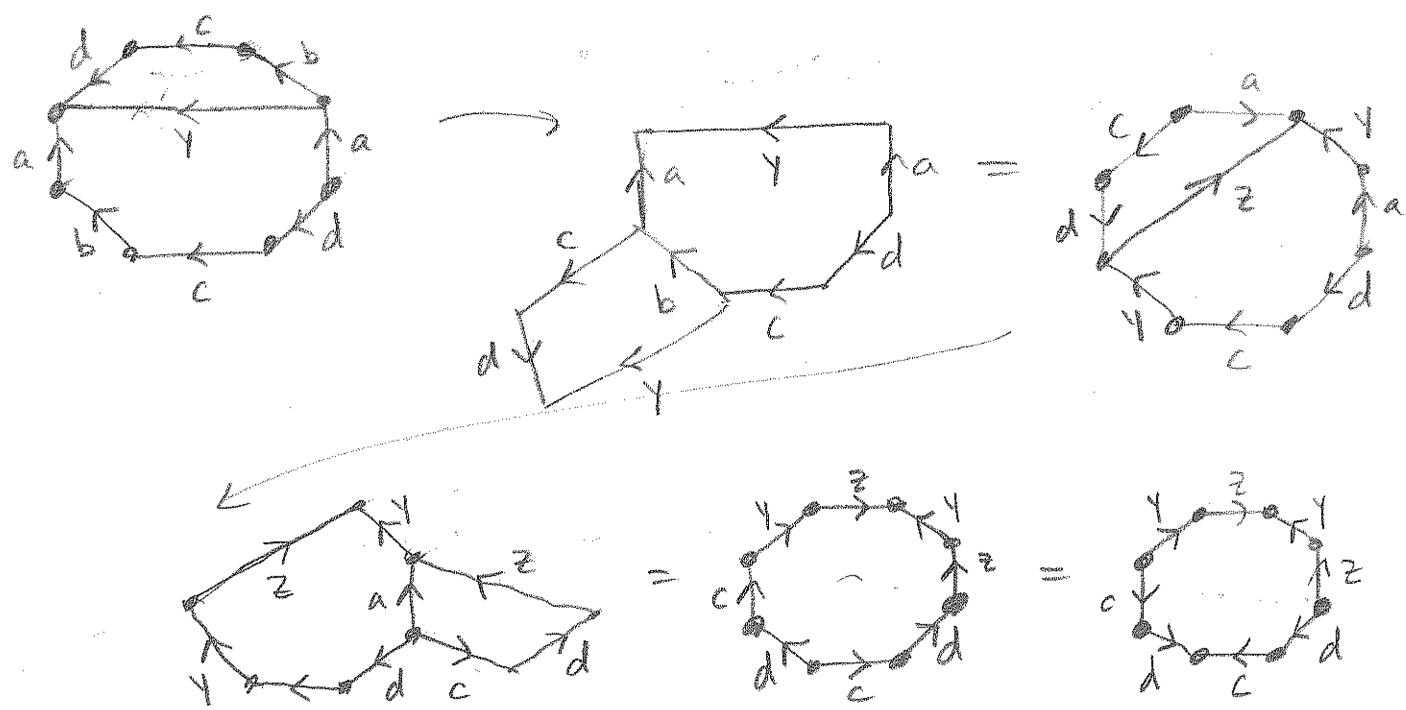
$aba^{-1}b^{-1}cdc^{-1}d^{-1}eeff$

(ii) $\mathbb{T}^2\sharp\mathbb{T}^2$ with two open disks removed (so the surface has two circle boundary components).



$aba^{-1}b^{-1}cdc^{-1}d^{-1}efe^{-1}ghg^{-1}$

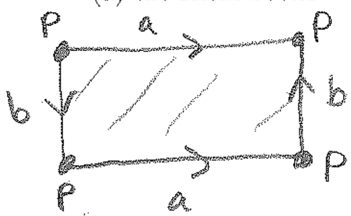
(b) Follow the algorithm outlined in Section 2.6 of the text to show that the surface obtained from an octagon by identifying edges according to the boundary word $abcda^{-1}b^{-1}c^{-1}d^{-1}$ is homeomorphic to $\mathbb{T}^2\sharp\mathbb{T}^2$.



$= \mathbb{T}^2 \# \mathbb{T}^2$

2. Determine the Euler-Poincaré characteristic $\chi(M)$ for each of the following spaces. You may use any cell structure to compute $\chi(M)$.

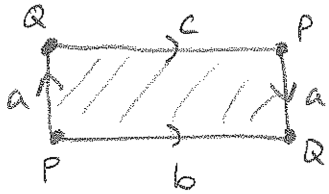
(a) the Klein bottle

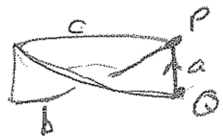


In the quotient, there is one vertex, two 1-cells, and one 2-cell

$\chi(M) = 1 - 2 + 1 = \boxed{0}$

(b) the Möbius band

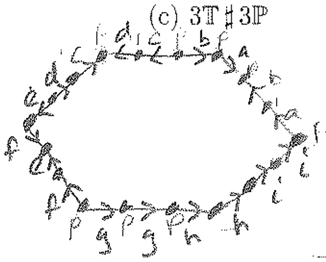




In the quotient, there are two vertices, three 1-cells, and one 2-cell.

$\chi(M) = 2 - 3 + 1 = \boxed{0}$

(c) $3\mathbb{T} \sharp 3\mathbb{P}$



There are 9 1-cells and one 2-cell. One can check that all 18 vertices get identified in the quotient.

So $\chi(M) = 1 - 9 + 1 = \boxed{-7}$

$\Big(\chi(3\mathbb{T} \# 3\mathbb{P}) = (\chi(3\mathbb{T}) - 1) + (\chi(3\mathbb{P}) - 1) = [(2 - 2\cdot 3) - 1] + [2 - 3 - 1] = (-5) + (-2) = -7 \Big)$

(d) $\mathbb{T}^2 \sharp \mathbb{T}^2$ with two open disks removed.

From 1(a)(ii), there are 8 1-cells, 1 2-cell and 3 vertices. Then $\chi(M) = 3 - 8 + 1 = \boxed{-4}$

$\big(\chi(M) = \chi(\mathbb{T}^2 \# \mathbb{T}^2) - 2 = (2 - 2\cdot 2) - 2 = -4. \big)$

(e) the Poincaré homology sphere, obtained from a dodecahedron (having 12 pentagonal faces meeting 3 at each vertex) by identifying opposite faces with a 72° counterclockwise twist.

(By the labelling in the figure I handed out in class, in the quotient there are 5 vertices (I–V), 10 edges (a–k missing j), 6 2-cells (A–F), and 1 3-cell. So $\chi(M^3) = 5 - 10 + 6 - 1 = 0$, same as S^3. In fact M^3 has the same homology as S^3.)