MAT 441 **Problem Set #4** Name SOLUTIONS
3/9/09 (due Wednesday 3/25/09)
15 points

Justify all answers. Unsupported answers may not receive full credit.

1. Suppose X is compact. Show that every infinite subset A of X has a limit point in X.

Hint: Argue by contradiction; note that, if A has no limit points, then A is closed (so $X - A$ is open).

Suppose $A \subseteq X$ has no limit point. Then every $x \in X$ has a neighborhood U_x such that $(U_x \cap A) - \{x\} = \phi$. Then $U_x \cap A = \phi$ if $x \notin A$, and $U_x \cap A = \{x\}$ if $x \in A$. The collection $\{U_x \mid x \in X\}$ is an open cover of X. Since X is compact, there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. Then $A = A \cap X = A \cap (\bigcup_{i=1}^{n} U_{x_i}) = \bigcup_{i=1}^{n} (A \cap U_{x_i}) \subseteq \bigcup_{i=1}^{n} \{x_i\} = \{x_1, \ldots, x_n\}$, so A is finite. Therefore, if A is an infinite subset of X, A has a limit point in X.

2. Prove: $X \times Y$ is connected if and only if X and Y are connected.

For the "if" direction, follow this outline: choose $(x_0, y_0) \in X \times Y$, and show, for every $(x, y) \in X \times Y$, there is a connected subspace of $X \times Y$ containing both (x_0, y_0) and (x, y). Use the lemma proved in class about unions of connected subspaces, and the fact that components are connected.

Lemma 1: If A and B are connected subspaces of a topological space W, and $A \cap B \neq \phi$, then $A \cup B$ is connected.
(proved in class)

Recall, the connected components of a space W are the equivalence classes of the equivalence relation $x \sim y$ iff x and y are contained in a connected subset of W.

Lemma 2: Components are connected.
pf: Let C be a component of W. Suppose $C = U \cup V$ with U and V open in C, nonempty, and $U \cap V = \phi$. Let $x \in U$ and $y \in V$. Then there is a connected subset A of W containing x and y. Clearly $A \subseteq C$. Then $A = (A \cap U) \cup (A \cap V)$, $A \cap U$ and $A \cap V$ are nonempty, disjoint, and open in A. Contradiction.

($\Longrightarrow$) $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous surjections, so $X \times Y$ connected implies X and Y are connected since the continuous image of a connected set is connected.

($\Longleftarrow$) Let $(x_0, y_0), (x_1, y_1) \in X \times Y$. Then $\{x_0\} \times Y$ is homeomorphic

(over) $\longrightarrow$

$\mathbb{R}^2 \not\approx \mathbb{R}$ and $\mathbb{R}^2 \not\approx [0,\infty)$ since $\mathbb{R}^2$ has no cut points. $\mathbb{R}^2 - \{(0,0)\}$ is homeo. to $S^1 \times \mathbb{R}$, hence is connected by problem 2. We have no method to distinguish $\mathbb{R}^2 - \{(0,0)\}$ from $\mathbb{R}^2$. Homeomorphism classes are indicated by labels below

3. Determine, if possible, which of the following spaces are homeomorphic. Give a short justification for your answers. (Some of these spaces are not distinguishable using connectedness and/or compactness arguments.)

$[0,1]$	$(0,1]$	$(0,\infty)$	$[0,\infty)$	$\mathbb{R}$	$\mathbb{R}-\{0\}$	S^1	$S^1-\{(1,0)\}$	$\mathbb{R}^2$	$\mathbb{R}^2-\{(0,0)\}$	S^2	$S^2-\{(0,0,1)\}$
A	B	C	B	C	D	E	C	F	G	H	F

Hint: Start with general topological properties. Then consider the number of "cut-points" and "non-cut-points" - a *cut-point* of a connected space is a point whose complement is not connected.

compact spaces: $[0,1]$, S^1, S^2

$(0,1] \approx [0,\infty)$
$(0,\infty) \approx \mathbb{R}$
and $\mathbb{R} \approx S^1 - \{(1,0)\}$ by stereographic projection

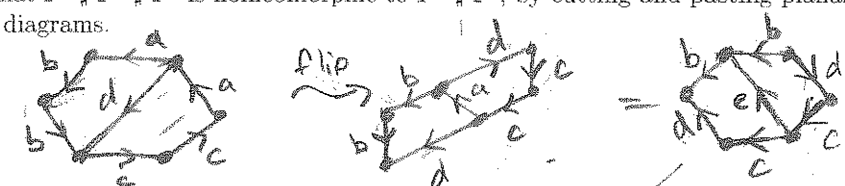
$\mathbb{R} \not\approx [0,\infty)$ since $[0,\infty)$ has one non-cut point and $\mathbb{R}$ has none.

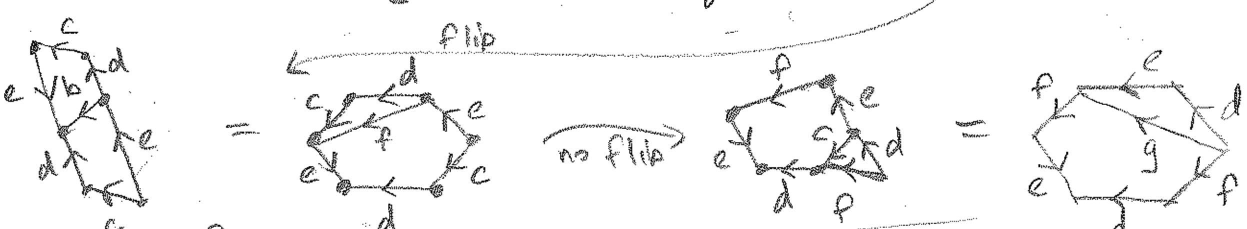
$\mathbb{R}^2 \approx S^2 - \{(0,0,1)\}$ by stereographic projection. $\mathbb{R}-\{0\}$ is the only space that is not connected.

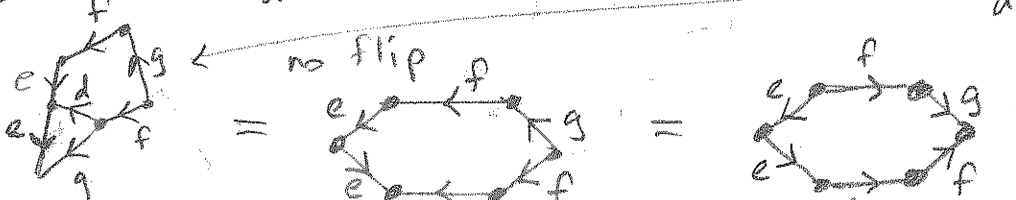
$[0,1] \not\approx S^1$ since $[0,1]$ has 2 non-cut points and S^1 has none. $S^1 \not\approx S^2$ since $S^1 - \{x,y\}$ is not connected for any two distinct points $x,y \in S^1$. But $S^2 - \{x,y\} \approx \mathbb{R}^2 - \{0\}$ is connected for any two distinct points in S^2. $[0,1] \not\approx S^2$ since $[0,1]$ has cut points and S^2 doesn't. ($S^2 - \{x\} \approx \mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ is connected by Problem 2 and Thm from class.)

4. (a) Show that $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ is homeomorphic to $T^2 \# \mathbb{P}^2$, by cutting-and-pasting planar identification diagrams.

$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 =$







$= T^2 \# \mathbb{P}^2$

(b) Use (a) and the classification theorem for surfaces to show that every surface is homeomorphic to S^2 or a connected sum of $m \geq 0$ tori and $n = 0, 1,$ or 2 projective planes.

$n\mathbb{P}^2 \approx T^2 \# (n-2)\mathbb{P}^2 \approx T^2 \# T^2 \# (n-4)\mathbb{P}^2 \approx \cdots \approx \left(\frac{n-1}{2}\right)T^2 \# \mathbb{P}^2$ (n odd)

or $\left(\frac{n-2}{2}\right)T^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ (n even)

Since every surface is $\approx S^2$, mT^2, or $n\mathbb{P}^2$, this proves the assertion.

(2) (continued) to Y, hence is connected. Similarly, $X \times \{y_1\}$ is connected. Since $(\{x_0\}\times Y) \cap (X \times \{y_1\}) = \{(x_0,y_1)\} \neq \emptyset$, $(\{x_0\}\times Y) \cup (X \times \{y_1\})$ is connected by Lemma 1. This set contains both (x_0,y_0) and (x_1,y_1), those points lie in the same component of $X\times Y$. Thus $X \times Y$ consists of a single component. Then $X\times Y$ is connected by Lemma 2.