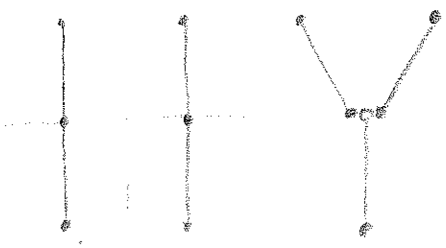
MAT 441 **Problem Set #3** Name SOLUTIONS

2/27/09 (due Friday 3/6/09)

15 points

Justify all answers. Unsupported answers may not receive full credit.

1. Let $X = \{-1, 1\} \times [-1, 1]$. (So X is a union of two disjoint line segments.) Let $\sim$ the the equivalence relation on X defined by $(-1, x) \sim (1, x)$ for $-1 \le x < 0$ (and $(x, y) \sim (x, y)$ for all $(x, y) \in X$). Show that the quotient space $X/\sim$ is not Hausdorff.



X $\qquad$ $X/\sim$

Let $x = [(-1,0)]$ and $y = [(1,0)]$ in $X/\sim$. Then $x \neq y$. Let V be an open nbhd. of x. Then $p^{-1}(V)$ is a saturated open nbhd of $(-1,0)$ in X, where $p: X \longrightarrow X/\sim$ is the canonical map. Then $\exists\, \varepsilon > 0$ such that $\{-1\} \times (-\varepsilon, \varepsilon) \subseteq p^{-1}(V)$. Similarly, if W is an open nbhd of y in $X/\sim$, then $\exists\, \delta > 0$ such that $\{1\} \times (-\delta, \delta) \subseteq p^{-1}(W)$. Let $y < 0$ with $y > -\delta$ and $y > -\varepsilon$. Then $(-1, y) \in p^{-1}(V)$ and $(1, y) \in p^{-1}(W)$. Then $p((-1,y)) \in V$ and $p((1,y)) \in W$. Since $y < 0$, $p((-1,y)) = p((1,y))$. Thus $V \cap W \neq \phi$. So x and y have no disjt. open nbhds, and $X/\sim$ is not Hausdorff.

2. Suppose A is a connected subspace of X. Prove that $\overline{A}$ is connected.

Suppose $\overline{A}$ is not connected. Then $\exists$ open subsets U and V of $\overline{A}$ with $U \neq \phi$, $V \neq \phi$, $U \cap V = \phi$, and $U \cup V = \overline{A}$. There are open sets U_1 and V_1 in X such that $U = U_1 \cap \overline{A}$ and $V = V_1 \cap \overline{A}$. Let $x \in U$. Then U_1 is an open nbhd of x, and $x \in \overline{A}$, so $U_1 \cap A \neq \phi$. Similarly $V_1 \cap A \neq \phi$. Let $U_2 = U_1 \cap A$ and $V_2 = V_1 \cap A$. Then U_2 and V_2 are open sets in A, $U_2 \neq \phi$, $V_2 \neq \phi$, $U_2 \cap V_2 = \phi$ (since $U_2 \cap V_2 \subseteq U \cap V$), and $U_2 \cup V_2 = A$. This contradicts the hypothes that A is connected.

3. Recall: a *path* in X from x to y is a continuous function $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. A space X is *path-connected* iff, for every $x, y \in X$, there is a path in X from x to y.

(a) Prove: if X is path-connected then X is connected.

Hint: Show that X has only one connected component.

Let $x, y \in X$. Then $\exists$ path $\gamma: [0,1] \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since $[0,1]$ is connected and γ is continuous, $\gamma([0,1])$ is connected. Thus x and y lie in a connected subspace of X. Since x and y were arbitrary, this shows that X has a single connected component. Since components are connected,* it follows that X is connected.

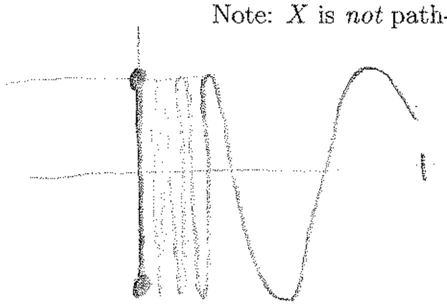
(b) Prove that the topologist's sine wave

$$X = \{(x, y) \in \mathbb{R}^2 \mid y = \sin(\frac{1}{x}),\ 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$$

is connected.

Hint: Use Problem 2.

Note: X is *not* path-connected, but that is harder to show.



Let $A = X \cap ((0,1] \times \mathbb{R})$. Then A is the graph of the continuous function $f : (0,1] \longrightarrow \mathbb{R}$; $f(x) = \sin(\frac{1}{x})$. Then A is homeomorphic to $(0,1]$. Hence A is connected. Since $X = \overline{A}$ it follows that X is connected, by Problem 2.

4. Suppose X is compact. Show that every infinite subset A of X has a limit point in X.

Hint: Argue by contradiction; note that, if A has no limit points, then A is closed (so $X - A$ is open).

postponed to HW #4.