MAT 441 **Problem Set #2** Name SOLUTIONS

1/30/09 (due Friday 2/6/09)

15 points

Justify all answers. Unsupported answers may not receive full credit.

Definition Let X be and Y be topological spaces, and let $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$. The *product topology* on $X \times Y$ is defined as follows: a subset $W \subseteq X \times Y$ is open if and only if, for every $(x,y) \in W$ there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x,y) \in U \times V \subseteq W$.

1. Show that $X \times Y$ is Hausdorff if both X and Y are Hausdorff.

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $(x_1, y_1) \neq (x_2, y_2)$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Suppose $x_1 \neq x_2$. Since X is Hausdorff, $\exists$ open sets U_1, U_2 in X with $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$. Then $U_1 \times Y$ and $U_2 \times Y$ are open in $X \times Y$, $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$, and $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \emptyset$. Similarly, if $y_1 \neq y_2$ then there are disjoint open sets $X \times V_1$ and $X \times V_2$ in $X \times Y$ containing (x_1, y_1) and (x_2, y_2), respectively (using the hypothesis that Y is Hausdorff). Thus $X \times Y$ is Hausdorff.

2. Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of the product $X \times X$, endowed with the product topology.

($\Longrightarrow$) Suppose X is Hausdorff. Let $(x,y) \in X \times X - \Delta$. Then $x, y \in X$ and $x \neq y$, so $\exists$ open nbhds. U, V of x and y, respectively, in X, with $U \cap V = \emptyset$. Then $U \times V$ is an open nbhd of (x,y) in $X \times X$. Claim $U \times V \subseteq X \times X - \Delta$. If not, then $(U \times V) \cap \Delta \neq \emptyset$, so $\exists$ $(p,p) \in \Delta$ with $(p,p) \in U \times V$, equivalently, $p \in U \cap V$. Since $U \cap V = \emptyset$ this cannot be. Thus $U \times V \subseteq X \times X - \Delta$. This shows $X \times X - \Delta$ is open, so Δ is closed. ($\Longleftarrow$) Let $x, y \in X$ with $x \neq y$. Then $(x,y) \in X \times X - \Delta$. Since Δ is closed by hypothesis, $X \times X - \Delta$ is open, and there are open subsets U and V in X such that $(x,y) \in U \times V$ and $U \times V \subseteq X \times X - \Delta$. Then $x \in U$ and $y \in V$, and, as above, $U \cap V = \emptyset$. Thus X is Hausdorff.

3. Let $f : X \to Y$ be a continuous function, with Y Hausdorff. The *graph* of f is the subset

$$\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$$

of $X \times Y$. Show that $\Gamma(f)$ is a closed subset of $X \times Y$.

We show $X \times Y - \Gamma(f)$ is open in $X \times Y$. Let $(x,y) \in X \times Y - \Gamma(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there are open sets U and V in Y such that $f(x) \in U$, $y \in V$, and $U \cap V = \phi$. Then $f^{-1}(U)$ is an open nbhd. of x in X, since f is continuous and $f(x) \in U$. Then $(x,y) \in f^{-1}(U) \times V$. Claim $f^{-1}(U) \times V \subseteq X \times Y - \Gamma(f)$. Indeed, if $(u,v) \in f^{-1}(U) \times V$ then $f(u) \in U$ and $v \in V$, so $f(u) \neq v$, since $U \cap V = \phi$, and so $(u,v) \notin \Gamma(f)$. This shows $X \times Y - \Gamma(f)$ is open, hence $\Gamma(f)$ is closed.

4. Suppose $f : X \to X$ is continuous, and X is Hausdorff. Show that the set of fixed points of f is closed in X.

A *fixed point* of $f : X \to X$ is a point $x \in X$ satisfying $f(x) = x$.

Let $F = \{x \in X \mid f(x) = x\}$. Let $x \in X - F$. Then $f(x) \neq x$. Since X is Hausdorff, there are open sets U and V in X with $x \in U$, $f(x) \in V$, and $U \cap V = \phi$. Since f is continuous $f^{-1}(V)$ is open in X, and $x \in f^{-1}(V)$ since $f(x) \in V$. Then $x \in U \cap f^{-1}(V)$ and $U \cap f^{-1}(V)$ is open in X. Claim $U \cap f^{-1}(V) \subseteq X - F$. Indeed, if $y \in U \cap f^{-1}(V)$, then $y \in U$ and $f(y) \in V$, so $y \neq f(y)$, since $U \cap V = \phi$. Thus $y \in X - F$. This shows $X - F$ is open, so F is closed.