MAT 441 **Problem Set #1** Name SOLUTIONS

1/21/09 (due Wednesday 1/28/09)

15 points

Justify all answers. Unsupported answers may not receive full credit.

Definition Let X be a topological space, let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X, and let $x \in X$. We say $\{x_n\}_{n=1}^{\infty}$ *converges* to x if and only if, for every open neighborhood[1] U of x in X, there exists $N \geq 1$ such that $x_n \in U$ for all $n \geq N$.

1. (a) The *Sierpinski space* is the topological space consisting of the set $X = \{0,1\}$ with the topology $\mathcal{T} = \{\emptyset, \{0\}, \{0,1\}\}$. Show that, (i) $\mathcal{T}$ is indeed a topology on X, and (ii) the constant sequence $x_n = 0$, $n = 1, 2, 3, \ldots$ converges to both 0 and 1 in $(X, \mathcal{T})$.

$\phi \in \mathcal{T}$ and $X \in \mathcal{T}$. If $U, V \in \mathcal{T}$, then $U \cap V = \phi$ (if $U = \phi$ or $V = \phi$) or $U \cap V = U$ or V (if $V = X$ or $U = X$, resp.), or $U \cap V = \{0\}$ (if $U = \{0\} = V$). If $\{U_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{T}$ then $\bigcup_{\lambda \in \Lambda} U_\lambda = \phi$ (if $U_\lambda = \phi \ \forall \lambda \in \Lambda$), or $\bigcup_{\lambda \in \Lambda} U_\lambda = X$ (if $U_\lambda = X$ for some $\lambda \in \Lambda$), or $\bigcup_{\lambda \in \Lambda} U_\lambda = \{0\}$ (if $U_\lambda = \{0\}$ for some $\lambda \in \Lambda$ and $U_\lambda \neq X$ for all $\lambda \in \Lambda$). Thus $\mathcal{T}$ is a topology.

(b) Now suppose X is a metric space with the metric topology. Show that a sequence $\{x_n\}_{n=1}^{\infty}$ in X can converge to at most one point of X.

Hint: Assume $\{x_n\}_{n=1}^{\infty}$ converges to both x and y, with $x \neq y$. Find a contradiction.

(1) cont'd. If $U = \{0\}$ or $\{0,1\}$ then $x_n \in U$ for all $n \geq 1$. Thus $\{x_n\}_{n=1}^{\infty}$ converges to 0. If $U = \{0,1\}$ then $x_n \in U$ for all $n \geq 1$, and thus $\{x_n\}_{n=1}^{\infty}$ converges to 1.

Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x and to y, and $x \neq y$. Let $\varepsilon = \frac{1}{2} d(x,y)$. Then $\exists\ N \geq 1$ such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$, and $\exists\ M \geq 1$ such that $x_n \in B(y, \varepsilon)$ for all $n \geq M$. Then, for $n = \max(M, N)$, $x_n \in B(x,\varepsilon) \cap B(y, \varepsilon)$. But $B(x,\varepsilon) \cap B(y,\varepsilon)$ is empty (proof below). Contradiction.

(Suppose $z \in B(x,\varepsilon) \cap B(y,\varepsilon)$. Then $d(x,y) \leq d(x,z) + d(y,z) < \varepsilon + \varepsilon = 2\varepsilon = d(x,y)$, contradiction.)

(c) Generalize your argument in (b) to Hausdorff topological spaces; a space X is *Hausdorff* if, for every $x, y \in X$ with $x \neq y$, there exists open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Suppose $\{x_n\}_{n=1}^{\infty}$ converges to x and y with $x \neq y$. Let U and V be open sets in X with $x \in U$, $y \in V$, and $U \cap V = \phi$. Then $\exists\ N \geq 1$ and $M \geq 1$ such that $x_n \in U\ \forall\ n \geq N$ and $x_n \in V\ \forall\ n \geq M$. Then, for $n = \max(M, N)$, $x_n \in U \cap V$. But $U \cap V = \phi$. Contradiction.

[1] An *open neighborhood* of a point x in a topological space X is an open subset U of X with $x \in U$.

Lemma Let X be a topological space, and $A \subseteq X$. Then $x \in \overline{A}$ if and only if, for every open neighborhood U of x in X, $U \cap A \neq \emptyset$.

2. (a) Prove the lemma above.

Let $B = \{x \in X \mid$ for every open nbhd. U of x, $U \cap A \neq \phi\}$. If $x \in A$ and U is a nbhd. of x, then $x \in U \cap A$, so $U \cap A \neq \phi$. Thus $x \in B$. If $x \in A'$ and U is an open nbhd of x, then $(U \cap A) - \{x\} \neq \phi$, so $U \cap A \neq \phi$. Thus $x \in B$. This shows $\overline{A} = A \cup A' \subseteq B$. Conversely, if $x \in B$ and $x \notin A$, and U is an open nbhd of x, then $U \cap A \neq \phi$ and $x \notin U \cap A$, so $U \cap A - \{x\} \neq \phi$. Thus $x \in A'$. This shows $B \subseteq A \cup A' = \overline{A}$. Therefore $\overline{A} = B$.

(b) Suppose $f\colon X \to Y$ is a continuous function, and $A \subseteq X$. Prove that $f(\overline{A}) \subseteq \overline{f(A)}$. Give an example to show that equality need not hold in general.

Let $y \in f(\overline{A})$. Then $y = f(a)$ for some $x \in \overline{A}$. Let V be an open nbhd of y in Y. Then $U = f^{-1}(V)$ is open in X, and $x \in U$ (since $f(x) = y \in V$). Since $x \in \overline{A}$ we have $U \cap A \neq \phi$ by (a). Let $a \in U \cap A$. Then $a \in f^{-1}(V)$, so $f(a) \in V$, and $a \in A$ so $f(a) \in f(A)$. Thus $V \cap f(A) \neq \phi$. Thus $y \in \overline{f(A)}$ by (a). Therefore $f(\overline{A}) \subseteq \overline{f(A)}$.

For a counterexample, let $X = \{0,1\}$ with the discrete topology, $Y = \{0,1\}$ with the indiscrete topology, $f\colon X \to Y$ the identity function, and $A = \{0\} \subseteq X$. Then $\overline{A} = \{0\}$ in X, but $\overline{f(A)} = \{0,1\}$ in Y, so $f(\overline{A}) = \{0\} \neq \{0,1\} = \overline{f(A)}$.