MAT 441
5/6/09
120 points

Final Exam

Name SOLUTIONS

1.(15) Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces, and let $X = \bigcup_{\lambda\in\Lambda} C_\lambda$. Let

$$\mathcal{T} = \{U \subseteq X \mid U \cap C_\lambda \text{ is open in } C_\lambda \text{ for every } \lambda \in \Lambda\}.$$

(a) Prove that $\mathcal{T}$ is a topology on X.[1]

For any $\lambda \in \Lambda$, $\phi \cap C_\lambda = \phi$ is open in C_λ, so $\phi \in \mathcal{T}$.
Let $\{U_\alpha \mid \alpha \in A\} \subseteq \mathcal{T}$, and $\lambda \in \Lambda$. Then $U_\alpha \cap C_\lambda$ is open in C_λ, so $\left(\bigcup_{\alpha\in A} U_\alpha\right) \cap C_\lambda = \bigcup_{\alpha \in A}(U_\alpha \cap C_\lambda)$ is open in C_λ.
Thus $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}$.
Let $U, V \in \mathcal{T}$, and $\lambda \in \Lambda$. Then $U \cap C_\lambda$ and $V \cap C_\lambda$ are open in C_λ, so $(U \cap V) \cap C_\lambda = (U \cap C_\lambda) \cap (V \cap C_\lambda)$ is open in C_λ. Thus $U \cap V \in \mathcal{T}$. Therefore $\mathcal{T}$ is a topology on X.

(b) Prove that the inclusion map $i_\lambda \colon C_\lambda \to X$ is continuous.[2]

Let $\lambda \in \Lambda$ and let U be an open subset of X.
Then $i_\lambda^{-1}(U) = U \cap C_\lambda$ is open in C_λ by definition.
Thus i_λ is continuous.

(c) Suppose $f \colon X \to Y$ is a function and the restriction $f|_{C_\lambda}$ is continuous for each $\lambda \in \Lambda$. Prove that f is continuous.

Let V be an open subset of Y, and let $U = f^{-1}(V)$.
Note that $U \cap C_\lambda = i_\lambda^{-1}(U) = (i_\lambda^{-1} \circ f^{-1})(V) = (f \circ i_\lambda)^{-1}(V) = (f|_{C_\lambda})^{-1}(V)$, which is open in C_λ by hypothesis, for each $\lambda \in \Lambda$. Thus U is open in X.
Therefore f is continuous.

[1] $\mathcal{T}$ is called the *weak topology* with respect to the family $\{C_\lambda \mid \lambda \in \Lambda\}$
[2] The inclusion $C_\lambda \to X$ need not be an embedding.

2.(20) Suppose $f\colon S^1 \to \mathbb{R}$ is a continuous function.

(a) Prove that f cannot be surjective.

S^1 is compact and f is continuous, so $f(S^1)$ is compact in $\mathbb{R}$. Since $\mathbb{R}$ is not compact, (see 4(b)), $f(S^1) \neq \mathbb{R}$, so f is not surjective.

(b) Prove that the image $f(S^1)$ of f is a closed, finite interval $[a, b]$.

Since S^1 is connected and f is continuous, $f(S^1)$ is connected. Then $f(S^1)$ is an interval. Since $f(S^1)$ is also compact, it is closed and bounded. The only closed, bounded intervals are the closed finite intervals $[a, b]$.

(c) Prove that f cannot be injective.

If f were injective, then $f: S^1 \to f(S^1)$ would be a continuous bijection from a compact space to a Hausdorff space, and hence f would be a homeomorphism. But $f(S^1) = [a, b]$ and S^1 is not homeomorphic to $[a, b]$. ($[a, b]$ has cut points and S^1 doesn't.)

3.(10) Express the surface $5\mathbb{T}^2\#3\mathbb{P}^2$ as a connected sum of projective planes, and use the result to find its Euler-Poincaré characteristic.

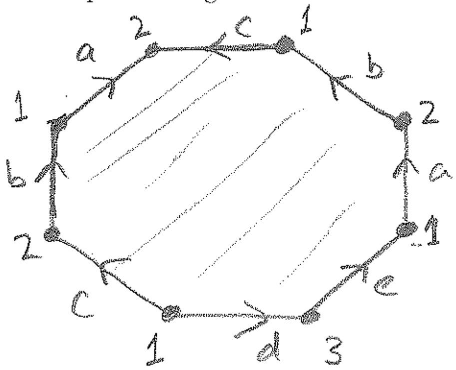
$\mathbb{T}^2 \# \mathbb{P}^2 \approx 3\mathbb{P}^2$, so

$$5\mathbb{T}^2 \# 3\mathbb{P}^2 \approx 4\mathbb{T}^2 \# 5\mathbb{P}^2 \approx 3\mathbb{T}^2 \# 7\mathbb{P}^2$$
$$\approx 2\mathbb{T}^2 \# 9\mathbb{P}^2$$
$$\approx \mathbb{T}^2 \# 11\mathbb{P}^2$$
$$\approx 13\mathbb{P}^2.$$

Then $\chi(5\mathbb{T}^2 \# 3\mathbb{P}^2) = \chi(13\mathbb{P}^2)$
$= 2 - 13 = \boxed{-11}$

6.(10) Consider the topological space given by identifying points on the boundary of the disk D^2 according to the boundary word $abca^{-1}b^{-1}c^{-1}de$.

(a) Find the number of vertices in the quotient space, in the cell structure inherited from the planar diagram.



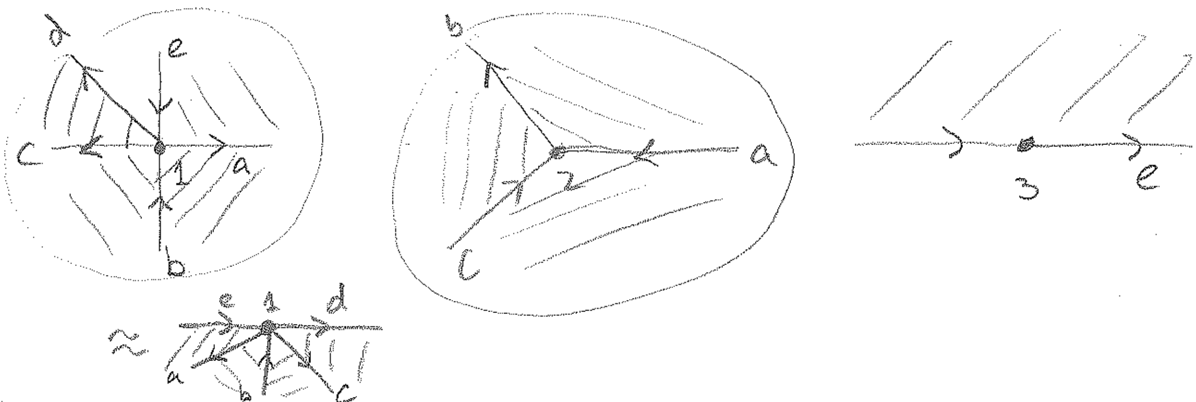
vertices are labelled w/ arabic numerals. Identifications are made to preserve incidence with edges.

There are 3 vertices.

(b) Compute the Euler-Poincaré characteristic of the quotient.

$$\chi(D^2/\sim) = 3 - 5 + 1 = -1$$

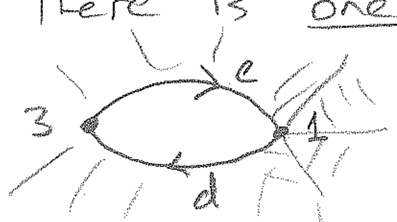
(c) Sketch small neighborhoods of the images of each of the vertices in the quotient space.



(c) Is the quotient space a topological surface, a surface-with-boundary, or neither? If the quotient space is a surface with or without boundary, identify the surface.

surface with boundary. (see 1 and 3 above).

There is <u>one</u> boundary component



attaching <u>one</u> 2-cell yields a surface without boundary whose Euler-Poincaré characteristic is

$$\chi(D^2/\sim) + 1 = (3-5+1)+1 = 0.$$

Also, $D^2/\sim$ is <u>orientable</u>, so $(D^2/\sim) \cup (\text{2-cell}) \approx \mathbb{T}^2$ and $D^2/\sim$ is a torus with a disk removed.

4.(10) Recall, a metric space (X, d) is *bounded* if there exists $x_0 \in X$ and $R \in \mathbb{R}$ such that $B_X(x_0, R) = X$.

(a) Give an example of a metric space which is bounded but not compact. (Justify your claims.)

Let $X = (-1, 1)$. Then $X = B_X(0, 1)$ so X is bounded. But X is not a closed subset of $\mathbb{R}$, (a Hausdorff space), so X is not compact.

(b) Prove: if (X, d) is a compact metric space, then (X, d) is bounded.

Let $x_0 \in X$. Consider the family $\mathcal{U} = \{B(x_0, n) \mid n \in \mathbb{N}\}$. Each $B(x_0, n)$ is open in X, and, for any $x \in X$, $x \in B(x_0, N)$ where $n = \lceil d(x_0, x) \rceil$ ($\lceil\ \rceil$ = the ceiling function). Thus $\mathcal{U}$ is an open cover of X. Since X is compact, there is a finite subcover $\{B(x_0, n_1), \cdots, B(x_0, n_p)\}$ of $\mathcal{U}$.

Let $R = \max\{n_1, \cdots, n_p\}$. Then $R \in \mathbb{R}$ and $B(x_0, n_i) \subseteq B(x_0, R)$ for $1 \le i \le p$. Then $X = B(x_0, n_1) \cup \cdots \cup B(x_0, n_p) = B(x_0, R)$, so X is bounded.

5.(10) Let X be an arbitrary topological space and $Y = \{0, 1\}$ with the discrete topology. Prove X is connected if and only if every continuous function $f \colon X \to Y$ is constant.

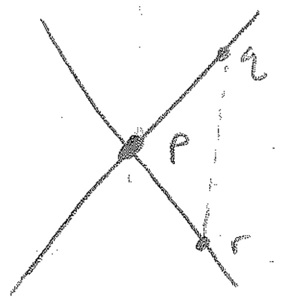
($\Longrightarrow$) Suppose X is connected, and $f: X \longrightarrow \{0,1\}$ is continuous. Then $f(X)$ is connected. Since $\{0,1\}$ is discrete, it is not connected, so $f(X)$ is a proper subset of $\{0,1\}$, hence f is constant.

$\Longleftarrow$ Suppose X is not connected. Then $\exists$ disjoint nonempty open subsets U and V of X with $X = U \cup V$. Define $f: X \longrightarrow [0,1]$ by $f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$. Then f is well-defined, since $U \cap V = \phi$, hence continuous by the gluing lemma, and f is not constant since both U and V are nonempty.

7.(10) A subset X of $\mathbb{R}^n$ is *starlike* with respect to $p \in X$ if, for every $q \in X$, the straight line segment $\overline{pq}$ is a subset of X.

(a) Give an example (by drawing a picture) of a starlike set which is not convex.

in $\mathbb{R}^2$:

$X =$ 

is starlike with respect to p but is not convex since $\overline{qr} \not\subseteq X$.

(b) Suppose X is starlike with respect to p. Show that the identity map $f\colon X \to X$, $f(x) = x$ is homotopic to the constant map $g : X \to X$, $g(x) = p$.

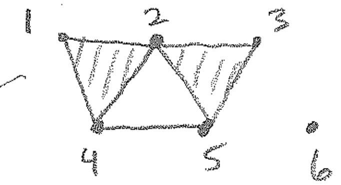
Let $H : X \times I \longrightarrow X$ be defined by $H(q,t) = (1-t)q + tp$. Since X is starlike w.r.t. p, $H(q,t) \in X$ for all $q \in X$, $t \in I$, so H is well-defined, and H is continuous by properties of vector multiplication and addition $H(q,0) = q = f(q)$ and $H(q,1) = p = g(q)$, so H is a homotopy of f to g.

(c) What, if anything, does (b) imply about the Euler-Poincaré characteristic of a star-like set? What about the betti numbers?

(b) shows that any starlike set X is contractible, hence has the same homotopy type as a point. Hence $\chi(X) = \chi(\{p\}) = 1$ and $\beta_i(X) = \beta_i(\{p\}) = \begin{cases} 1 & i=0 \\ 0 & i>0. \end{cases}$

8.(20) Let K be the simplicial complex with vertex set $\{1,2,3,4,5,6\}$ having maximal simplices 124, 235, and 45~~6~~, and 6.

(a) Determine the number of simplices in K, and the Euler-Poincaré characteristic of K.



Since any subset of a simplex in K is in K,

$$K = \{1,2,3,4,5,6,12,14,23,24,25,35,45,124,235\}.$$

$$\chi(K) = 6 - 7 + 2 = 1$$

(b) Sketch a geometric realization $|K|$ of K.

(c) Write the matrices of the boundary maps $\partial_2 \colon C_2(K) \to C_1(K)$ and $\partial_1 \colon C_1(K) \to C_0(K)$.

$C_2(K) \cong \mathbb{R}^2$, $C_1(K) \cong \mathbb{R}^7$, $C_0(K) \cong \mathbb{R}^6$

$$\partial_2 = \begin{array}{c} \\ 12 \\ 14 \\ 23 \\ 24 \\ 25 \\ 35 \\ 45 \end{array} \begin{array}{c} \begin{array}{cc} 124 & 235 \end{array} \\ \left[\begin{array}{cc} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{array}\right] \end{array}$$

$$\partial_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 12 & 14 & 23 & 24 & 25 & 35 & 45 \end{array} \\ \left[\begin{array}{ccccccc} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(d) Find the betti numbers of K. ~~Show that $|K|$ has the same betti numbers as S^1. You may use the computer to calculate ranks. (You need not compute the betti numbers of S^1.)~~

rank $\partial_2 = 2$ so nullity $\partial_2 = 2 - 2 = 0$

rank $\partial_1 = 4$ so nullity $\partial_1 = 7 - 4 = 3$

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$\beta_2(K) = \text{nullity}(\partial_2) = 0$

$\beta_1(K) = \text{nullity}(\partial_1) - \text{rank}(\partial_2) = 3 - 2 = 1.$

$\beta_0(K) = \dim(C_0) - \text{rank}(\partial_1) = 6 - 4 = 2.$

(note: $|K|$ has $2 = \beta_0(K)$ connected components.)

Please answer the following questions, and refrain from writing your name on this page.

Are you working towards a baccalaureate degree from the Department of Mathematics and Statistics?
No.
If yes, please circle your degree program: *B.S.* *B.S.Ed.* *B.S. Extended*

If B.S. Extended, please circle your emphasis area:
Mathematics *Statistics* *Applied Mathematics* *Actuarial Science*

9.(15) For each topological space below, indicate whether or not the space is (a) Hausdorff, (b) metrizable, (c) connected, and (d) compact.

(i) The Sierpinski space (the set $\{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$).

(a) not Hausdorff, hence (b) not metrizable
(c) connected and (d) compact.

(ii) the set of integers $\mathbb{Z}$ with the discrete topology.

Hausdorff, metrizable (as a subspace of $\mathbb{R}$),
not connected, not compact.

(iii) the open interval $(0, 1)$.

Hausdorff, metrizable, connected, not compact.

(iv) the half-open interval $[0, \infty)$.

Hausdorff, metrizable, connected, not compact.

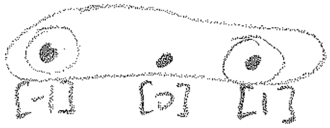
(v) the quotient of the real line $\mathbb{R}$ by the relation $x \sim y$ iff $|x| = |y|$.

$\longrightarrow [0, \infty)$

$x \longmapsto |x|$

$\mathbb{R}/\sim \;\approx [0, \infty)$,
Hausdorff, metrizable,
connected, not
compact.

(vi) the quotient of the real line $\mathbb{R}$ by the relation $x \sim y$ iff $x = y$ or $xy > 0$.



not Hausdorff,
not metrizable,
connected, compact.

(vii) the set of integers with the cofinite topology.

not Hausdorff, not metrizable, connected,
compact

(viii) the real line $\mathbb{R}$ with the half-open interval topology.

Hausdorff, not metrizable (difficult-to-prove
fact stated in lecture)
not connected, not compact.