MAT 441 **Exam 3** Name SOLUTIONS
4/17/09
70 points

Provide some justification for all nontrivial claims, to receive full credit. You may use any theorems stated in lecture without proof.

1.(20) (a) Show that every integer n less than or equal to 2 occurs as the Euler-Poincaré characteristic of some compact surface without boundary.

The standard cell structure on $g\mathbb{T}^2$ has 1 vertex, $2g$ 1-cells, and 1 2-cell, so $\chi(g\mathbb{T}^2) = 1 - 2g + 1 = 2 - 2g$. Then, if N is an even integer, $N < 2$, then $\frac{1}{2}(2-N) = g$ is positive, and $\chi(g\mathbb{T}^2) = 2 - 2g = 2-(2-N) = N$. The standard cell structure on $g\mathbb{P}^2$ has 1 vertex, g 1-cells, and 1 2-cell, so $\chi(g\mathbb{P}^2) = 1 - g + 1 = 2 - g$. Then, if $N < 2$, $g = 2-N$ is positive and $\chi(g\mathbb{P}^2) = 2-(2-N) = N$. Since $\chi(S^2) = 2$, this completes the argument.

(b) Up to homeomorphism, how many different compact surfaces S are there with given Euler-Poincare characteristic $\chi(S)$, in case

(i) $\chi(S)$ is even.

If $\chi(S) = 2$ there is one.
If $\chi(S) = N < 2$ is even, there are two: $S = g\mathbb{P}^2$ with $g = 2 - \chi(S)$, and $S = g\mathbb{T}^2$ with $g = \frac{1}{2}(2 - \chi(S))$. (see part (a).)

(ii) $\chi(S)$ is odd.

There is one: $S = g\mathbb{P}^2$ with $g = 2 - \chi(S)$.

(c) Make a list of all compact surfaces with non-negative Euler-Poincaré characteristic.

$\chi(S^2) = 2$

$\chi(\mathbb{P}^2) = 1$

$\chi(\mathbb{T}^2) = 0$

$\chi(\mathbb{P}^2 \# \mathbb{P}^2) = 0$

Klein bottle.

That is all.

2.(20) Each expression below is meant to identify the identification pattern for a planar identification diagram for a topological space. In each case

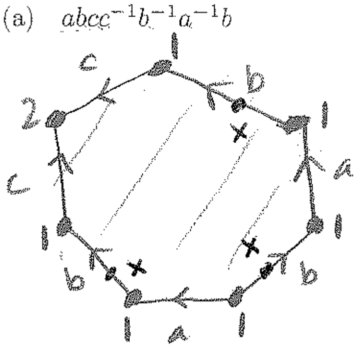
(i) determine whether the quotient space is a topological surface (without boundary);

(ii) if the quotient space is a surface, determine whether the that surface is orientable or not.

(iii) compute the Euler-Poincaré characteristic of the quotient space (whether or not the quotient space is a surface).

(iv) if the quotient space is a surface, identify it according to the classification theorem for surfaces (i.e., as a connected sum of $\mathbb{T}^2$'s and/or $\mathbb{P}^2$'s. Hint: Use (iii)

(a) $abcc^{-1}b^{-1}a^{-1}b$

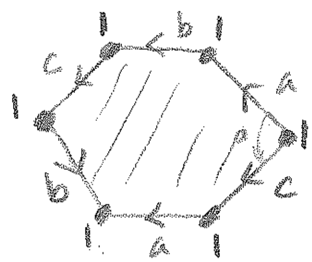


Not a surface; a neighborhood of x in the quotient looks like

not $\approx \mathbb{R}^2$

There are 3 1-cells and 1 2-cell; one also finds that there are 2 vertices, so the Euler-Poincaré characteristic is $2-3+1=\boxed{0}$

(b) $abcba^{-1}c^{-1}$



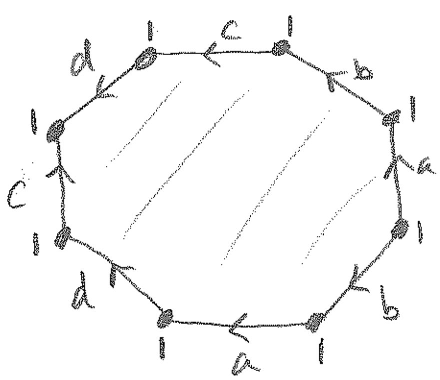
The quotient is a surface, since edges are identified in pairs. There is one vertex in the quotient

, which also has a nbhd $\approx \mathbb{R}^2$. (See also Prop. 2.6.2)

The surface is not orientable since b is traversed twice with consistent orientation (so the surface contains a Möbius band).

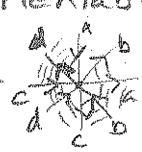
The Euler-Poincaré characteristic is $1-3+1=-1$ so the surface is $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$ or $\mathbb{T}^2 \# \mathbb{P}^2$

(c) $abcdc^{-1}d^{-1}a^{-1}b^{-1}$



The quotient is a surface because edges are identified in pairs. It is orientable because the edges in each pair are oppositely oriented.

There is 1 vertex



, 4 edges, and 1 2-cell; $\chi = 1-4+1=-2$ so the surface is $\mathbb{T}^2 \# \mathbb{T}^2$.

3.(15) (a) Suppose S' is a compact surface with boundary, obtained from a compact surface S by removing k open disks (with disjoint closures)[1]. Express $\chi(S')$ in terms of $\chi(S)$ and k.

Choose a triangulation of S having k disjoint 2-simplices* and remove their interiors to form S'. Then removing those 2-simplices yields a triangulation of S'. Then $\chi(S) = \chi(S') + k$, so $\boxed{\chi(S') = \chi(S) - k.}$

* Note: this can be accomplished by repeatedly subdividing:

(b) Make a list of all compact surfaces with boundary having non-negative Euler-Poincare characteristic. $k \geq 1$ and $\chi(S') \geq 1$, so

$1 \leq \chi(S') + k = \chi(S) \leq 2$ (using problem 1), so there are these possibilities:

$\chi(S)$	S	k	S'
2	S^2	1	D^2 (disk)
2	S^2	2	$S^1 \times I$ (cylinder)
1	$\mathbb{P}^2$	1	$\mathbb{P}^2 - D^2$ (Möbius band)

(c) Identify the surfaces pictured below (from the Math/Stat Department website `http://www.cefns.nau.edu/Academic/Math/researchInterests/`) by calculating the Euler-Poincaré characteristic and counting the number of boundary components. Construct a cell decomposition by inserting vertices along the boundary components and 1-cells in the surface separating the twists or crossings. These surfaces are orientable - if you look at the colored pictures on the web page you'll see that the surfaces have two sides, one green and one blue.

see additional sheet

Note: All surfaces are connected and orientable.

Seifert surfaces by Jack van Wijk, TU Eindhoven

Surface # 1: # boundary components $= 1$; $f_0 = 12$, $f_1 = 12 + 6 = 18$, $f_2 = 5$, $\chi(S') = 12 - 18 + 5 = -1$, so $\chi(S) = -1 + 1 = 0$. S is orientable, so $S = \mathbb{T}^2$ and $S' = \mathbb{T}^2 - D^2$.

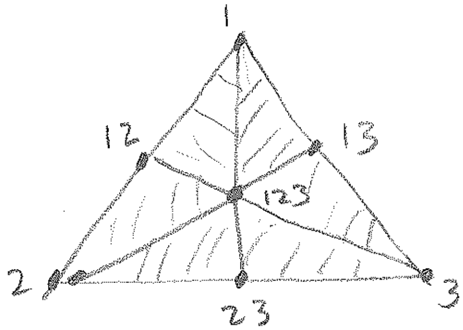
Surface # 3: # boundary comps $= 3$; $f_0 = 16$, $f_1 = 16 + 8 = 24$, $f_2 = 7$, $\chi(S') = 16 - 24 + 7 = -1$

so $\chi(S) = -1 + 3 = 2$, $S = S^2$, $S' = S^2 - 3$ disks

[1] It can be shown that every compact surface with boundary can be obtained this way.

see additional sheet for surface # 2

4.(5) Let V be the set of nonempty subsets of $\{1,2,3\}$. Let K be the abstract simplicial complex with vertex set V whose simplices are the subsets σ of V which are *linearly ordered*, that is, for every $A, B \in \sigma$, $A \subseteq B$ or $B \subseteq A$. Show that K determines a triangulation of the standard simplex Δ^2.

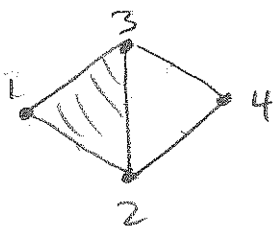


vertices 1, 2, 3, 12, 13, 23, 123

1-simplices $\{1,12\}$, $\{1,13\}$, $\{2,12\}$, $\{2,23\}$, $\{3,13\}$, $\{3,23\}$, $\{1,123\}$, $\{2,123\}$, $\{3,123\}$, $\{12,123\}$, $\{13,123\}$, $\{23,123\}$.

2-simplices $\{1,12,123\}$, $\{2,12,123\}$, $\{1,13,123\}$, $\{3,13,123\}$, $\{2,23,123\}$, $\{3,23,123\}$

5.(10) Let K be the abstract simplicial complex $K = \{1,2,3,4,12,13,23,24,34,123\}$[2]. Compute the simplicial homology of K (with real coefficients).



$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$
$$\quad\ \ \| \qquad\quad \|$$
$$0 \longrightarrow \mathbb{R}^1 \xrightarrow{\partial_2} \mathbb{R}^5 \xrightarrow{\partial_1} \mathbb{R}^4 \longrightarrow 0$$

$$\partial_2 = \begin{array}{c c} & 123 \\ \begin{array}{c} 12 \\ 13 \\ 23 \\ 24 \\ 34 \end{array} & \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \quad \text{rank } \partial_2 = 1, \text{ nullity } \partial_2 = 1-1 = 0$$

$$\Longrightarrow \boxed{H_2(K) = \ker(\partial_2) = 0}$$

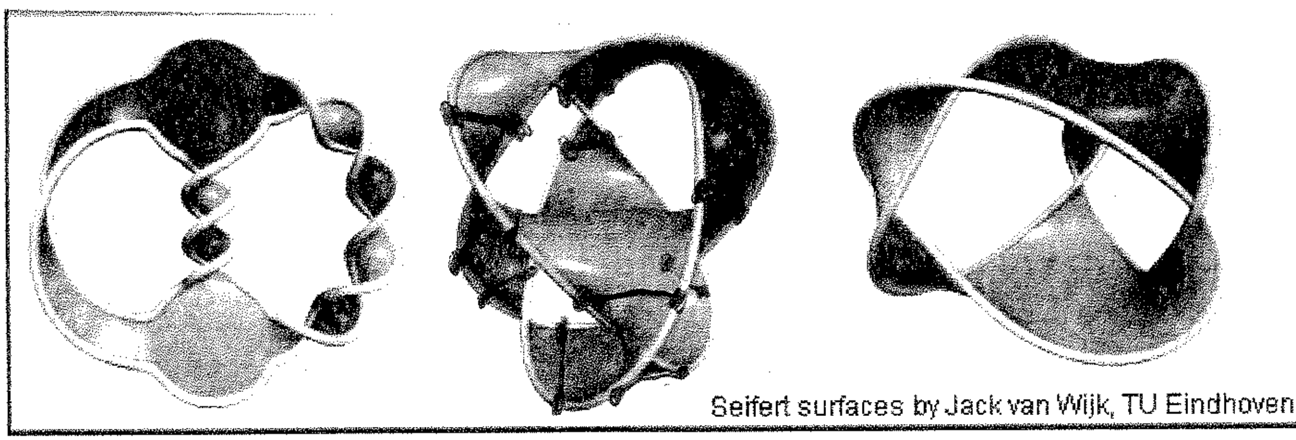
$$\partial_1 = \begin{array}{c c} & \begin{array}{ccccc} 12 & 13 & 23 & 24 & 34 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \underset{\text{row equiv}}{\sim} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $\partial_1 = 3$, nullity $\partial_1 = 5 - 3 = 2$

so $\boxed{H_1(K) = \ker \partial_1 / \operatorname{im} \partial_2 \cong \mathbb{R}^2/\mathbb{R} \cong \mathbb{R}^1}$ and $\boxed{H_0(K) \cong \mathbb{R}^4/\operatorname{im}\partial_1 = \mathbb{R}^4/\mathbb{R}^3 \cong \mathbb{R}}$

[2]We are using the standard shorthand, denoting sets of numbers by listing their elements without commas or set braces.

$$\text{so} \quad H_i(K) = \begin{cases} \mathbb{R} & \text{if } i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$



surface # 2 : # boundary components = 3

$f_0 = 20$, $f_1 = 20 + 10 = 30$, $f_2 = 9$

$\chi(S') = 20 - 30 + 9 = -1$

$\chi(S) = -1 + 3 = 2$, so $S = S^2$

$S' = S^2 - 3$ disks