MAT 441 **Exam 2** Name SOLUTIONS
3/13/09
65 points

Provide some justification for all nontrivial claims, to receive full credit. You may use any theorems stated in lecture without proof.

1.(12) (a) Suppose X is an infinite set, endowed with the discrete topology. Prove that X is not compact.

For each $x \in X$, $\{x\}$ is an open set in X. Then $\mathcal{U} = \{\{x\} \mid x \in X\}$ is an open cover of X. Since X is infinite, $\mathcal{U}$ has no finite subcover.

(b) Let Y be the set of integers, with the cofinite topology[1] Prove that Y is compact.

Let $\mathcal{U}$ be an open cover of Y. Let $U_0 \in \mathcal{U}$, with $U_0 \neq \phi$. Then $Y - U_0$ is finite, say $Y - U_0 = \{y_1, \ldots, y_n\}$. For each i, $1 \leq i \leq n$, there is an element $U_i \in \mathcal{U}$ with $y_i \in U_i$. Then $\{U_0, U_1, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$. Thus Y is compact.

2.(8) Suppose X is a topological space and there is a non-constant continuous function $f\colon X \to Y$ to a discrete space Y. Show that X is not connected.

Let $x \in X$. Let $U = \{f(x)\}$ and $V = Y - \{f(x)\}$. Then $f^{-1}(U) \neq \phi$, and, since f is not constant, $f^{-1}(V) \neq \phi$. Since Y is discrete, U and V are open in Y. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Since $U \cap V = \phi$, $f^{-1}(U) \cap f^{-1}(V) = \phi$, and since $U \cup V = Y$, $f^{-1}(U) \cup f^{-1}(V) = X$. Thus X is not connected.

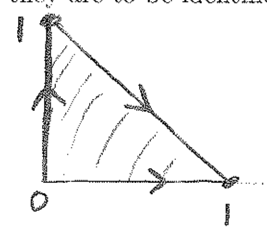
[1]A subset is open iff it is empty or its complement is finite.

3.(8) Recall, a continuous function $f: X \to Y$ is a *closed mapping* if, for every closed subset A of X, the image $f(A)$ is closed in Y. Prove: if $f: X \to Y$ is a continuous function, X is compact, and Y is Hausdorff, then f is a closed mapping.

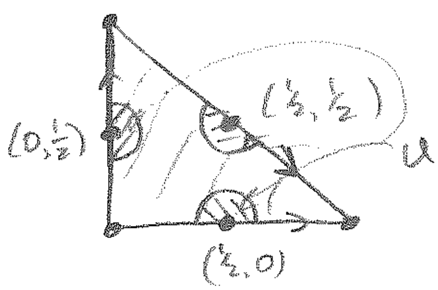
Let A be closed in X. Since X is compact, A is compact. Since f is continuous and A is compact, $f(A)$ is compact. Then, since Y is Hausdorff, $f(A)$ is closed.

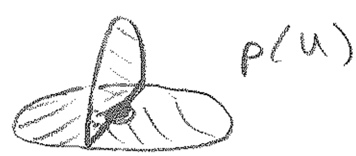
4 ~~5~~.(25 ~~30~~) Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 - x \text{ and } x \geq 0\}$. Let $\sim$ be the equivalence relation on X generated by $(x, 0) \sim \cancel{(x, 1)}$ $(0, x)$ and $(x, 0) \sim (x, 1 - x)$. Let $p: X \to X/\sim$ be the quotient map. ($X/\sim$ is called the *dunce cap*.)

(a) Sketch an "identification diagram" for $\sim$, that is, a picture of X with arrows on the edges to indicate how they are to be identified in $X/\sim$.

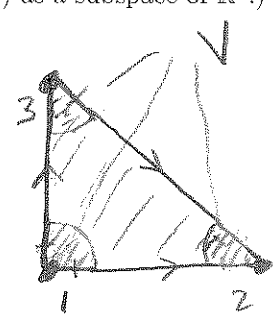


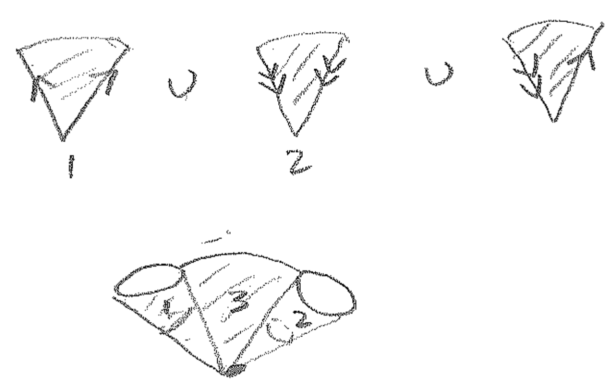
(b) Sketch a picture of X and an open subset U of X (with $U \neq X$) such that $p(U)$ is an open neighborhood of $[(0.5, 0)]$ in $X/\sim$. Then sketch or describe $p(U)$.



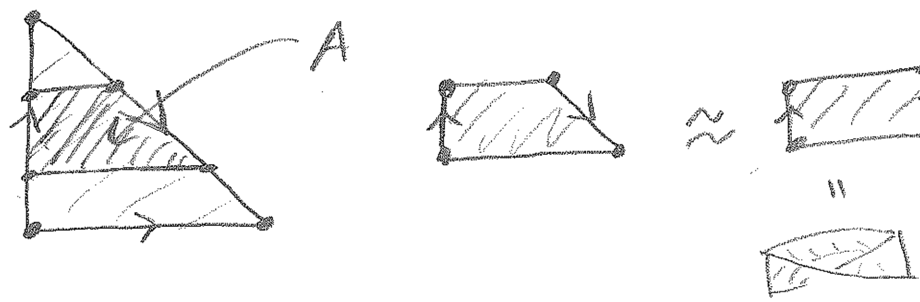


(c) Sketch a picture of X and an open subset V of X (with $V \neq X$) such that $p(V)$ is an open neighborhood of $[(0, 0)]$ in $X/\sim$. Then sketch or describe $p(V)$. (Choose V so that you can easily sketch $p(V)$ as a subspace of $\mathbb{R}^2$.)





(d) Sketch a picture of X and a closed subspace A of X such that $p(A)$ is homeomorphic to the Möbius band.



(e) Is $X/\sim$ a topological surface?

No, neither $[(\frac{1}{2},0)]$ nor $[(0,0)]$ has an open nbhd. homeomorphic to $\mathbb{R}^2$.

5.(~~10~~ 6) Suppose X is connected and compact, and $\sim$ is an equivalence relation on X. Is $X/\sim$ necessarily connected? Is $X/\sim$ necessarily compact? (Justify your answers.)

$p: X \longrightarrow X/\sim$ is a continuous surjection, so, if X is connected, $p(X) = X/\sim$ is connected, and, if X is compact then $p(X) = X/\sim$ is compact.

6.(~~10~~ 6) Suppose $f\colon X \to Y$ is a closed surjection. (See Problem 3 for definition.) Prove that f is an identifcation map.

f is continuous by hypothesis. Suppose $U \subseteq Y$ and $f^{-1}(U)$ is open in X. Then $X - f^{-1}(U)$ is closed in X. Then $f(X - f^{-1}(U))$ is closed in Y. Since f is surjective, $f(X - f^{-1}(U)) = Y - U$. Thus $Y - U$ is closed, and U is open. Thus f is an identification map.