MAT 441 **Exam 1 (Take-home)** Name SOLUTIONS

2/13/07 (due Wednesday 2/18)

60 points

Take-home exam rules: you may consult any published texts, but are not to consult with any human resource except the instructor by any means. Hints are available from the instructor. No theorems other than those we discussed during the course are needed to solve these problems - those results can be used freely. All other nontrivial statements should be supported.

1.(16) (a) Suppose X is a T_1 space[1] and $A \subseteq X$. Show that $(A')' \subseteq A'$.

Note: There is a hint on the web page.

Lemma If $U \subseteq X$ is open and $K \subseteq X$ is closed, then $U - K$ is open. proof: $X - K$ is open and $U - K = U \cap (X - K)$. Thus $U - K$ is open.

Now let $x \in (A')'$ and let U be an open nbhd. of x. Then $U \cap A' \neq \phi$. Let $y \in U \cap A'$. If $x = y$ let $V = U$; if $x \neq y$ let $V = U - \{x\}$. Then V is an open nbhd of y and $V \subseteq U$. Since $y \in A'$ we conclude $A \cap (V - \{y\}) \neq \phi$. Since $x \notin A \cap (V - \{y\})$, we conclude that $A \cap (U - \{x\}) \neq \phi$, hence $x \in A'$. $\square$

(b) Show that any finite T_1 space is discrete.

Since any finite union of closed sets is closed, any finite subset of a T_1 space is closed. So, if X is a finite T_1 space, every subset is closed, hence every subset is open, hence X is discrete.

(c) Let $X = \{0, 1, 2\}$ with the topology $\mathcal{T} = \{\emptyset, \{0, 1\}, \{1\}, \{1, 2\}, X\}$, and let $A = \{1\}$. Show that $(A')' \neq A'$.

$A' = \{0, 2\}$ since every open nbhd of each of 0 and 2 meets A in a point different of 0 and 2, but not so for 1.

$(A')' = \{0, 2\}' = \phi \neq A'$.

(d) Prove: if X is a T_1 space, $A \subseteq X$, and $x \in A'$, then every open neighborhood of x contains infinitely many points of A.

Note: There is a hint on the web page.

Suppose X is T_1, $A \subseteq X$, $x \in A'$, and U is an open nbhd of x. Suppose $U \cap A$ is finite. Then $(U \cap A) - \{x\}$ is finite, hence closed. Then $V = U - (U \cap A - \{x\})$ is open, and $x \in V$, but $A \cap (V - \{x\}) = A \cap (U - (U \cap A)) - \{x\} = \phi$, contradicting the assumption that $x \in A'$.

Thus $U \cap A$ is infinite.

[1] That is, $\{x\}$ is closed for every $x \in X$.

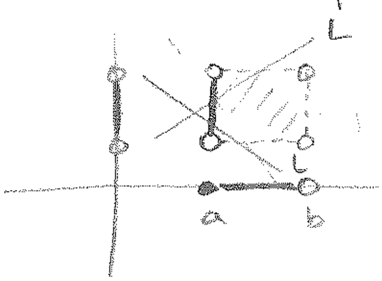
2.(12) Let $\mathbb{R}$ denote the set of reals with the standard topology, and $\mathbb{R}_\ell$ the set of reals with the half-open interval topology.[2]

(a) Let f be the identity function, $f(x) = x$ for all real numbers x. Determine whether $f: \mathbb{R} \to \mathbb{R}_\ell$ and/or $f: \mathbb{R}_\ell \to \mathbb{R}$ are continuous, and prove your answer.

Let $-\infty < a < b < \infty$. Then $(a,b) = \bigcup_{n \geq 1} [a + 1/n, b)$, hence (a,b) is open in $\mathbb{R}_\ell$. It follows that any open set in $\mathbb{R}$ is open in $\mathbb{R}_\ell$. Thus $f: \mathbb{R}_\ell \longrightarrow \mathbb{R}$ is continuous.
On the other hand, $[a,b)$ is not open in $\mathbb{R}$, so $f: \mathbb{R} \longrightarrow \mathbb{R}_\ell$ is not continuous.

(b) Let $X = \mathbb{R}_\ell \times \mathbb{R}$, with the product topology. Let L be a straight line in X, with the subspace topology. Determine conditions under which L is homeomorphic to $\mathbb{R}$, or to $\mathbb{R}_\ell$, or neither.

A basic open set in $\mathbb{R}_\ell \times \mathbb{R}$ has the form $[a,b) \times (c,d)$



If L is not vertical, then L is homeomorphic to $\mathbb{R}_\ell$, since $L \cap ([a,b) \times (c,d))$ can be a half-open interval (but not a single point).
If L is vertical then L is homeomorphic to $\mathbb{R}$.

(c) Let $x, y \in \mathbb{R}_\ell$, with $x \neq y$. Prove that there are open subsets U and V of $\mathbb{R}_\ell$ with $x \in U, y \in V$, $U \cap V = \emptyset$, and $U \cup V = \mathbb{R}_\ell$. (A space with this property is said to be *totally disconnected.*)

Assume without loss of generality that $x < y$.
From part (a), (a,y) is open for all $a < y$, so
$U = (-\infty, y) = \bigcup_{a<y} (a,y)$ is open, and it contains x.
Similarly, $V = [y, \infty) = \bigcup_{b>y} [y,b)$ is open, and it contains x.
$U \cap V = (-\infty, y) \cap [y, \infty) = \emptyset$, and $U \cup V = (-\infty, y) \cup [y, \infty)$
$= \mathbb{R}_\ell$.

[2]So $\mathbb{R}_\ell$ has basis consisting of the half-open intervals $[a, b)$ with $-\infty < a < b < \infty$.

alternative proof: X is homeomorphic to the subspace $X \times \{y\}$ of $X \times Y$, and any subspace of a Hausdorff space is Hausdorff. So $X \times \{y\}$ is Hausdorff, hence X is Hausdorff.

3.(12) (a) Let X and Y be nonempty topological spaces. Suppose $X \times Y$ is Hausdorff. Prove that X is Hausdorff.

Let $x, x' \in X$ with $x \neq x'$. Choose $y \in Y$. (Since $Y \neq \phi$ we can do that.) Then $(x,y) \neq (x',y)$. Since $X \times Y$ is Hausdorff, there are disjoint open nbhds W and W' of (x,y) and (x',y), respectively. Then $\exists$ open sets U, U' in X and V, V' in Y with $(x,y) \in U \times V \subseteq W$ and $(x',y) \in U' \times V' \subseteq W'$. Then $x \in U$, $x' \in U'$, and $U \cap U' = \phi$. Thus X is Hausdorff.

(b) Let $p \colon X \times Y \to X$ be the canonical projection, $p(x,y) = x$. Show that p is an open map.

Let W be an open set in $X \times Y$.
Let $x \in p(W)$. Then
$\exists$ $y \in Y$ with $(x,y) \in W$. Then $\exists$ open sets U in X, V in Y with $(x,y) \in U \times V \subseteq W$.
Then $x \in U$ and $U \subseteq p(W)$. It follows that $p(W)$ is open.

(c) Find an example to show that $p \colon X \times Y \to X$ need not be a closed map, that is, it need not map closed sets to closed sets.

Note: You may find it convenient to use the result of HW #2.3

Let $X = Y = \mathbb{R}$. Let $K = \{(x,y) \in \mathbb{R}^2 \mid x = \tan^{-1}(y)\}$.
Then K is closed in $\mathbb{R}^2$: K is the image under the homeomorphism $(x,y) \longmapsto (y,x)$ of the graph of the continuous function $f(x) = \tan^{-1}(x)$, which is closed by HW 2.3.
But $p(K) = (-\pi/2, \pi/2)$, which is not closed in $\mathbb{R}$.

4.(5) Let $f : X \to Y$ be a continuous function. Let $\Gamma(f) \subseteq X \times Y$ be the graph of f, defined by $\Gamma(f) = \{(x,y) \in X \times Y \mid y = f(x)\}$, considered as a subspace of $X \times Y$. Show that $\Gamma(f)$ is homeomorphic to X.

Let $h : X \longrightarrow \Gamma(f)$ be defined by $h(x) = (x, f(x))$.
h is continuous since each component function is continuous.
Let $p : \Gamma(f) \longrightarrow X$ be defined by $p(x,y) = x$.
p is continuous because it is the restriction to Γ of the (continuous) projection $X \times Y \longrightarrow X$.

Moreover, $(p \circ h)(x) = p(x, f(x)) = x$, and
$(h \circ p)(x,y) = (x, f(x)) = (x,y)$ for $(x,y) \in \Gamma(f)$.

Thus h and p are inverse bijections, so X is homeomorphic to $\Gamma(f)$.

5.(15) Parts (a) and (b) of this exercise shows that for general (non-metrizable) topological spaces a limit point of a subset A need not be the limit of a sequence of points in A.

Let $X = \mathbb{R}$ with the *co-countable* (or "countable complement") topology: a subset U is open iff $U = \emptyset$ or $\mathbb{R} - U$ is countable[3] It is easy to check that this is indeed a topology on X (e.g., using the axioms for closed sets).

(a) Suppose $A \subseteq X$ is an uncountable set (for instance, $A = [0,1]$). Show that every point $x \in X$ is a limit point of A.

Hint: A subset of a countable set must be countable.

Let $x \in X$. Let U be an open neighborhood of x. Then $X - U$ is countable, so $(X - (U - \{x\})) = (X - U) \cup \{x\}$ is countable. Then $A \not\subseteq (X - (U - \{x\}))$, so $A \cap (U - \{x\}) \neq \emptyset$. Thus $x \in A'$.

(b) Show that no sequence $(x_n)_{n=1}^{\infty}$ in X converges, unless for some $N \geq 1$, $x_n = x_N$ for all $n \geq N$.

Let $x \in X$ and $U = (X - \{x_n \mid n \geq 1\}) \cup \{x\}$. Then $X - U \subseteq \{x_n \mid n \geq 1\}$ so $X - U$ is countable. Thus U is an open neighborhood of x. If $\{x_n\}_{n=1}^{\infty}$ converges to x then $\exists\ N \geq 1$ such that $x_n \in U$ for all $n \geq N$, which implies $x_n = x$ for all $n \geq N$.

(c) Suppose X is a metric space, $A \subset X$, and $x \in A'$. Prove that there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ that converges to x.

For each $n \geq 1$, the open ball $B(x, 1/n)$ is an open nbhd of x. Since $x \in A'$, $A \cap (B(x, 1/n) - \{x\}) \neq \emptyset$, so we can choose $x_n \in A \cap (B(x, 1/n) - \{x\})$. Claim $\{x_n\}_{n=1}^{\infty}$ converges to x: if U is an open nbhd of x then $\exists\ \varepsilon > 0$ with $B(x, \varepsilon) \subseteq U$. Then $\exists\ N \geq 1$ with $\frac{1}{N} < \varepsilon$, and then, for every $n \geq N$, $x_n \in B(x, \frac{1}{n}) \subseteq B(x, 1/N) \subseteq B(x, \varepsilon) \subseteq U$.

[3] Recall, a set C is countable iff it is finite or there is a bijection $\mathbb{N} \to C$.