MAT 441 **Final Exam** Name ______________________

5/6/09

120 points

1.(15) Let $\{C_\lambda \mid \lambda \in \Lambda\}$ be a family of topological spaces, and let $X = \bigcup_{\lambda\in\Lambda} C_\lambda$. Let

$$\mathcal{T} = \{U \subseteq X \mid U \cap C_\lambda \text{ is open in } C_\lambda \text{ for every } \lambda \in \Lambda\}.$$

(a) Prove that $\mathcal{T}$ is a topology on X.[1]

(b) Prove that the inclusion map $i_\lambda \colon C_\lambda \to X$ is continuous.[2]

(c) Suppose $f \colon X \to Y$ is a function and the restriction $f|_{C_\lambda}$ is continuous for each $\lambda \in \Lambda$. Prove that f is continuous.

[1] $\mathcal{T}$ is called the *weak topology* with respect to the family $\{C_\lambda \mid \lambda \in \Lambda\}$

[2] The inclusion $C_\lambda \to X$ need not be an embedding.

2.(20) Suppose $f\colon S^1 \to \mathbb{R}$ is a continuous function.

(a) Prove that f cannot be surjective.

(b) Prove that the image $f(S^1)$ of f is a closed, finite interval $[a, b]$.

(c) Prove that f cannot be injective.

3.(10) Express the surface $5\mathbb{T}^2\#3\mathbb{P}^2$ as a connected sum of projective planes, and use the result to find its Euler-Poincaré characteristic.

4.(10) Recall, a metric space (X, d) is *bounded* if there exists $x_0 \in X$ and $R \in \mathbb{R}$ such that $B_X(x_0, R) = X$.

(a) Give an example of a metric space which is bounded but not compact. (Justify your claims.)

(b) Prove: if (X, d) is a compact metric space, then (X, d) is bounded.

5.(10) Let X be an arbitrary topological space and $Y = \{0, 1\}$ with the discrete topology. Prove X is connected if and only if every continuous function $f: X \to Y$ is constant.

6.(10) Consider the topological space given by identifying points on the boundary of the disk D^2 according to the boundary word $abca^{-1}b^{-1}c^{-1}de$.

(a) Find the number of vertices in the quotient space, in the cell structure inherited from the planar diagram.

(b) Compute the Euler-Poincaré characteristic of the quotient.

(c) Sketch small neighborhoods of the images of each of the vertices in the quotient space.

(c) Is the quotient space a topological surface, a surface-with-boundary, or neither? If the quotient space is a surface with or without boundary, identify the surface.

7.(10) A subset X of $\mathbb{R}^n$ is *starlike* with respect to $p \in X$ if, for every $q \in X$, the straight line segment $\overline{pq}$ is a subset of X.

(a) Give an example (by drawing a picture) of a starlike set which is not convex.

(b) Suppose X is starlike with respect to p. Show that the identity map $f\colon X \to X,\ f(x) = x$ is homotopic to the constant map $g : X \to X,\ g(x) = p$.

(c) What, if anything, does (b) imply about the Euler-Poincaré characteristic of a star-like set? What about the betti numbers?

8.(20) Let K be the simplicial complex with vertex set $\{1, 2, 3, 4, 5, 6\}$ having maximal simplices 124, 235, and 45.

(a) List all the simplices in K, and determine the Euler-Poincaré characteristic of K.

(b) Sketch a geometric realization $|K|$ of K.

(c) Write the matrices of the boundary maps $\partial_2 \colon C_2(K) \to C_1(K)$ and $\partial_1 \colon C_1(K) \to C_0(K)$.

(d) Show that $|K|$ has the same betti numbers as S^1. You may use the computer to calculate ranks. (You need not compute the betti numbers of S^1.)

Please answer the following questions, and refrain from writing your name on this page.

Are you working towards a baccalaureate degree from the Department of Mathematics and Statistics?
No.
If yes, please circle your degree program: *B.S.* *B.S.Ed.* *B.S. Extended*

If B.S. Extended, please circle your emphasis area:
Mathematics *Statistics* *Applied Mathematics* *Actuarial Science*

9.(15) For each topological space below, indicate whether or not the space is (a) Hausdorff, (b) metrizable, (c) connected, and (d) compact.

(i) The Sierpinski space (the set $\{0, 1\}$ with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$).

(ii) the set of integers $\mathbb{Z}$ with the discrete topology.

(iii) the open interval $(0, 1)$.

(iv) the half-open interval $[0, \infty)$.

(v) the quotient of the real line $\mathbb{R}$ by the relation $x \sim y$ iff $|x| = |y|$.

(vi) the quotient of the real line $\mathbb{R}$ by the relation $x \sim y$ iff $x = y$ or $xy > 0$.

(vii the set of integers with the cofinite topology.

(viii) the real line $\mathbb{R}$ with the half-open interval topology.