MAT 441 **Exam 2** Name ____________________

3/13/09

65 points

Provide some justification for all nontrivial claims, to receive full credit. You may use any theorems stated in lecture without proof.

1.(12) (a) Suppose X is an infinite set, endowed with the discrete topology. Prove that X is not compact.

(b) Let Y be the set of integers, with the cofinite topology[1] Prove that Y is compact.

2.(8) Suppose X is a topological space and there is a non-constant continuous function $f\colon X \to Y$ to a discrete space Y. Show that X is not connected.

[1] A subset is open iff it is empty or its complement is finite.

3.(8) Recall, a continuous function $f\colon X \to Y$ is a *closed mapping* if, for every closed subset A of X, the image $f(A)$ is closed in Y. Prove: if $f\colon X \to Y$ is a continuous function, X is compact, and Y is Hausdorff, then f is a closed mapping.

4.(30) Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1 - x \text{ and } x \ge 0\}$. Let $\sim$ be the equivalence relation on X generated by $(x, 0) \sim (0, x)$ and $(x, 0) \sim (x, 1 - x)$. Let $p\colon X \to X/\sim$ be the quotient map. ($X/\sim$ is called the *dunce cap*.)

(a) Sketch an "identification diagram" for $\sim$, that is, a picture of X with arrows on the edges to indicate how they are to be identified in $X/\sim$.

(b) Sketch a picture of X and an open subset U of X (with $U \neq X$) such that $p(U)$ is an open neighborhood of $[(0.5, 0)]$ in $X/\sim$. Then sketch or describe $p(U)$.

(c) Sketch a picture of X and an open subset V of X (with $V \neq X$) such that $p(V)$ is an open neighborhood of $[(0, 0)]$ in $X/\sim$. Then sketch or describe $p(V)$. (Choose V so that you can easily sketch $p(V)$ as a subspace of $\mathbb{R}^2$.)

(d) Sketch a picture of X and a closed subspace A of X such that $p(A)$ is homeomorphic to the Möbius band.

(e) Is $X/\sim$ a topological surface?

5.(10) Suppose X is connected and compact, and $\sim$ is an equivalence relation on X. Is $X/\sim$ necessarily connected? Is $X/\sim$ necessarily compact? (Justify your answers.)

6.(10) Suppose $f\colon X \to Y$ is a closed surjection. (See Problem 3 for definition.) Prove that f is an identification map.