MAT 441 **Exam 1 (Take-home)** Name ________________

2/13/07 (due Wednesday 2/18)

60 points

Take-home exam rules: you may consult any published texts, but are not to consult with any human resource except the instructor by any means. Hints are available from the instructor. No theorems other than those we discussed during the course are needed to solve these problems - those results can be used freely. All other nontrivial statements should be supported.

1.(16) (a) Suppose X is a T_1 space[1] and $A \subseteq X$. Show that $(A')' \subseteq A'$.

Note: There is a hint on the web page.

(b) Show that any finite T_1 space is discrete.

(c) Let $X = \{0,1,2\}$ with the topology $\mathcal{T} = \{\emptyset, \{0,1\}, \{1\}, \{1,2\}, X\}$, and let$A = \{1\}$. Show that $(A')' \neq A'$.

(d) Prove: if X is a T_1 space, $A \subseteq X$, and $x \in A'$, then every open neighborhood of x contains infinitely many points of A.

Note: There is a hint on the web page.

[1]That is, $\{x\}$ is closed for every $x \in X$.

2.(12) Let $\mathbb{R}$ denote the set of reals with the standard topology, and $\mathbb{R}_\ell$ the set of reals with the half-open interval topology.[2]

(a) Let f be the identity function, $f(x) = x$ for all real numbers x. Determine whether $f\colon \mathbb{R} \to \mathbb{R}_\ell$ and/or $f\colon \mathbb{R}_\ell \to \mathbb{R}$ are continuous, and prove your answer.

(b) Let $X = \mathbb{R}_\ell \times \mathbb{R}$, with the product topology. Let L be a straight line in X, with the subspace topology. Determine conditions under which L is homeomorphic to $\mathbb{R}$, or to $\mathbb{R}_\ell$, or neither.

(c) Let $x, y \in \mathbb{R}_\ell$, with $x \neq y$. Prove that there are open subsets U and V of $\mathbb{R}_\ell$ with $x \in U, y \in V$, $U \cap V = \emptyset$, and $U \cup V = \mathbb{R}_\ell$. (A space with this property is said to be *totally disconnected.*)

[2] So $\mathbb{R}_\ell$ has basis consisting of the half-open intervals $[a, b)$ with $-\infty < a < b < \infty$.

3.(12) (a) Let X and Y be nonempty topological spaces. Suppose $X \times Y$ is Hausdorff. Prove that X is Hausdorff.

(b) Let $p\colon X \times Y \to X$ be the canonical projection, $p(x, y) = x$. Show that p is an open map.

(c) Find an example to show that $p\colon X \times Y \to X$ need not be a closed map, that is, it need not map closed sets to closed sets.

Note: You may find it convenient to use the result of HW #2.3

4.(5) Let $f : X \to Y$ be a continuous function. Let $\Gamma(f) \subseteq X \times Y$ be the graph of f, defined by $\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$, considered as a subspace of $X \times Y$. Show that $\Gamma(f)$ is homeomorphic to X.

5.(15) Parts (a) and (b) of this exercise shows that for general (non-metrizable) topological spaces a limit point of a subset A need not be the limit of a sequence of points in A.

Let $X = \mathbb{R}$ with the *co-countable* (or "countable complement") topology: a subset U is open iff $U = \emptyset$ or $\mathbb{R} - U$ is countable[3] It is easy to check that this is indeed a topology on X (e.g., using the axioms for closed sets).

(a) Suppose $A \subseteq X$ is an uncountable set (for instance, $A = [0, 1]$). Show that every point $x \in X$ is a limit point of A.

Hint: A subset of a countable set must be countable.

(b) Show that no sequence $(x_n)_{n=1}^{\infty}$ in X converges.

(c) Suppose X is a metric space, $A \subset X$, and $x \in A'$. Prove that there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq A$ that converges to x.

[3] Recall, a set C is countable iff it is finite or there is a bijection $\mathbb{N} \to C$.