MAT 441C
2/27/13 (due Wednesday 3/6)
25 points

HW #4

Name ______________________

Justify all answers. Unsupported answers may not receive full credit.

1. (a) Suppose X is a topological space, and A is a connected subspace of X. Suppose $A \subseteq B \subseteq \overline{A}$. Prove B is connected.

   (b) Prove that the topologist's sine wave $X = \{(x\sin(\frac{1}{x}) \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$ is connected.

2. Let X be a topological space. Let $\sim$ be the relation on X defined by $x \sim y$ if and only off there is a connected subspace A of X containing both x and y.

   (a) Prove that $\sim$ is an equivalence relation.

   Note: It may be convenient to use Exercise I from the web page.

   (b) Show that the equivalence classes of $\sim$ are connected. These equivalence classes are called the *connected components* of X.

   Hint: The equivalence class $[x]$ is the union of the family of all connected subspaces of X containing x.

3. Show that the connected components of a space X are necessarily closed, but need not be open.

Hint: For the first part, use 1(a). For the second, think about convergent sequences in $\mathbb{R}$.

4. Suppose X and Y are connected spaces. Prove $X \times Y$ is connected.

Hint: Prove that $X \times Y$ has only one connected component.

5. A topological space X is *totally disconnected* if its connected components are singletons. Suppose X has the property that, for every two distinct points $x, y \in X$, there are disjoint nonempty open sets U and V in X such that $x \in U$, $y \in V$, and $U \cup V = X$. Prove X is totally disconnected.