MAT 441C **HW #2** Name ______________________

1/28/13 (due Monday 2/4)

25 points

Justify all answers. Unsupported answers may not receive full credit.

1. A topological space is is called a T_1 *space* if $\{x\}$ is a closed set for each $x \in X$.

   (a) Prove: X is a T_1 space if and only if, for every $x, y \in X$, if $x \neq y$, then there exists an open neighborhood U of y such that $x \notin U$.

   (b) Prove: if X is a finite T_1 space, then X is discrete.

   (c) Prove: $\mathbb{Z}$ with the cofinite topology is a T_1 space but is not Hausdorff.

   (d) Suppose X is T_1 and $A \subseteq X$. (i) Prove that A' is a closed set, and (ii) find an example to show that the statement may be false if X is not T_1. (Recall A' denotes the set of limit points of A.)

   Hint for (ii): Try $X = \{1, 2, 3\}$.

2. Let X and Y be topological spaces. By definition, a function $f\colon X \to Y$ is *continuous* if $f^{-1}(U)$ is open in X for every open subset U of Y. Recall $f^{-1}(U) = \{x \in X \mid f(x) \in U\}$.

(a) Suppose $f\colon X \to Y$ is continuous and $A \subseteq X$. (i) Prove that $f(\overline{A}) \subseteq \overline{f(A)}$, and (ii) give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and a closed subset A of $\mathbb{R}$ for which $f(A)$ is not closed. (Hence, equality need not hold in (i).)

(b) Suppose $f\colon X \to X$ is continuous, and X is Hausdorff. Show that the set of fixed points of f is closed in X. Here $x \in X$ is a *fixed point* of f iff $f(x) = x$.