MAT 137 **HW #7** Name SOLUTIONS

04/19/09 (due Thursday 04/23/09)

10 points

Show work or otherwise justify your answers. Unsupported answers (i.e. calculator output) will not receive full credit. You may check your answers with a calculator or computer.

1. (a) Find the degree-two Taylor polynomial $P_2(x)$ at $a = 0$ for the function $f(x) = \sqrt{1+3x}$. centered at $a = 0$.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $f^{(n)}(0)/n!$ |
|---|---|---|---|
| 0 | $(1+3x)^{1/2}$ | 1 | 1 |
| 1 | $\frac{3}{2}(1+3x)^{-1/2}$ | $3/2$ | $3/2$ |
| 2 | $-\frac{9}{4}(1+3x)^{-3/2}$ | $-9/4$ | $-9/8$ |

$$\boxed{P_2(x) = 1 + \frac{3}{2}x - \frac{9}{8}x^2}$$

(b) Use the Taylor-Lagrange derivative formula for the error to show that $|f(x) - P_2(x)|$ is less than 0.002 for $|x| < 0.1$.

$$|f(x) - P_2(x)| = \left| \frac{f^{(3)}(c)}{3!} x^3 \right|$$

$f^{(3)}(x) = \frac{81}{8}(1+3x)^{-5/2}$ is monotonic decreasing on the interval $-0.1 < x < 0.1$ so it attains its largest value at the left endpoint $x = -0.1$

So $|f^{(3)}(c)| \leq \left| \frac{81}{8}(1+3(-0.1))^{-5/2} \right|$

$$= \frac{81}{8}(0.7)^{-5/2} = \frac{81}{8}\left(\frac{7}{10}\right)^{-5/2}$$

$$= \frac{81}{8}\left(\frac{10}{7}\right)^{5/2} < \frac{81}{8}\left(\frac{10}{7}\right)^3 = \frac{81000}{2744} < 30$$

Then $|f(x) - P_2(x)| = \dfrac{|f^{(3)}(c)|}{3!}|x|^3$

$$< \frac{30}{6}(0.1)^3 < 0.002 .$$

2. Find the degree-three Taylor polynomial $P_3(x)$ at $a = 1$ for the function $f(x) = x^3 + x + 1$ and show that $P_3(x) = f(x)$.

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ | $f^{(n)}(1)/n!$ |
|---|---|---|---|
| 0 | $x^3+x+1$ | 3 | 3 |
| 1 | $3x^2+1$ | 4 | 4 |
| 2 | $6x$ | 6 | 3 |
| 3 | $6$ | 6 | 1 |

$$\boxed{P_3(x) = 3 + 4(x-1) + 3(x-1)^2 + 1(x-1)^3}$$

Check: $P_3(x) = 3 + 4x - 4 + 3x^2 - 6x + 3 + x^3 - 3x^2 + 3x - 1$

$= (3-4+3-1) + (4-6+3)x + (3-3)x^2 + x^3$

$= 1 + x + x^3 = f(x)$ ✓.

3. (a) Find the Taylor series about $a = 0$ for the function $f(x) = e^{-x^2}$, by substituting into the series for $e^x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{so} \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

or $$\boxed{e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}}$$

OR $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

so $$e^{-x^2} = 1 - x^2 + x^4/2! - x^6/3! + \cdots$$

(b) Find a power series which converges for all $x$, whose sum $f(x)$ is an antiderivative of $e^{-x^2}$.

Integrate term-by-term:

$$\int e^{-x^2}\,dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(n!)(2n+1)}}$$