MAT 137 **HW #6** Name SOLUTIONS

04/03/09 (due Friday 04/10/09)

10 points

Show work or otherwise justify your answers. Unsupported answers (i.e. calculator output) will not receive full credit. You may check your answers with a calculator or computer.

1. Determine which series converge. Use convergence tests to prove your answers.

(a) $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ $\boxed{\text{diverges}}$

$0 < \frac{1}{2n} = \frac{1}{n+n} < \frac{1}{n+\ln(n)}$ because $\ln(n) < n$, and $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series, $p \le 1$), so $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ diverges by the comparison test.

$\boxed{\text{OR}}$

$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+\ln(n)}} = \lim_{n\to\infty} \frac{n+\ln(n)}{n} = \lim_{n\to\infty}\left(1 + \frac{\ln(n)}{n}\right) = 1 \neq 0$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ diverges by the limit comparison test.

(b) $\sum_{n=1}^{\infty} \frac{n^2\, 2^n}{n!}$ Hint: Ratio test. $\boxed{\text{converges}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2\, 2^{n+1}}{(n+1)!}}{\frac{n^2\, 2^n}{n!}} = \frac{(n+1)^2\, 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2\, 2^n} = \frac{(n+1)^2}{n^2} \cdot \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \left(\frac{2^n \cdot 2}{2^n}\right)\left(\frac{n!}{(n+1)(n!)}\right) = \left(\frac{n+1}{n}\right)^2 (2)\left(\frac{1}{n+1}\right)$$

$\xrightarrow[n\to\infty]{} (1)^2(2)(0) = 0 = \rho < 1$, so the series converges by the ratio test.

(c) $\sum_{n=2}^{\infty} \frac{1}{n\,[\ln(n)]^2}$

$\left(\frac{1}{n(\ln(n))^2}\right)_{n=2}^{\infty}$ is decreasing, and $\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx =$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x(\ln(x))^2}\,dx = \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{1}{u^2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_{x=2}^{x=b}$$

$u = \ln(x)$
$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} -\frac{1}{\ln(x)}\Big|_{x=2}^{x=b} = \lim_{b\to\infty} -\frac{1}{\ln(b)} + \frac{1}{\ln(2)}$$

$$= -0 + \frac{1}{\ln(2)}.$$

The integral $\boxed{\text{converges}}$, so the series converges by the integral test.

(d) $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$ Hint: Comparison test.

$\ln(n) < \sqrt{n}$ ($\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = 0$ by L'Hôpital)

so $\frac{1}{\ln(n)} > \frac{1}{\sqrt{n}}$, so

$\frac{1}{(\ln(n))^2} > \frac{1}{(\sqrt{n})^2} = \frac{1}{n} > 0$, and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^2}$ $\boxed{\text{diverges}}$ by the comparison test.

(e) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^3+1}}$ $\boxed{\text{diverges}}$ by the n^{th} term test

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+1}} = \lim_{n\to\infty} \sqrt{\frac{n^4}{n^3+1}} = \sqrt{\lim_{n\to\infty} \frac{n^4}{n^3+1}}$$ since $\sqrt{\ }$ is continuous

by L'Hôpital's rule $= \sqrt{\lim_{n\to\infty} \frac{4n^3}{3n^2}} = \sqrt{\lim_{n\to\infty} \frac{4}{3} n} = \infty \neq 0$.

2. Determine which series converge conditionally and which converge absolutely. Use convergence tests to prove your answers.

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$: $\sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{\ln(n)}{n}\right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges

since $\frac{\ln(n)}{n} > \frac{1}{n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

But $\left(\frac{\ln(n)}{n}\right)_{n=1}^{\infty}$ is decreasing $\left(\frac{d}{dx}\left(\frac{\ln(x)}{x}\right) = \frac{x(\frac{1}{x}) - \ln(x)}{x^2} < 0 \text{ for } x > 1\right)$

and $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, so $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln(n)}{n}$ converges by the alternating series test. So the series $\boxed{\text{converges conditionally}}$.

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^3}{2^n}$

$\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ converges by the ratio test: $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3}{2^{n+1}} \Big/ \frac{n^3}{2^n} = \frac{(n+1)^3}{2^{n+1}} \cdot \frac{2^n}{n^3} = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{2}$

so $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1^3 \cdot \frac{1}{2} = \frac{1}{2} < 1$. So the original series $\boxed{\text{converges absolutely}}$.

(c) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\sqrt{n^3+1}}$

$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ diverges, since $\frac{n}{\sqrt{n^3+1}} = \sqrt{\frac{n^2}{n^3+1}} > \sqrt{\frac{n^2}{2n^3}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{n}}$

and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. But $\left(\frac{n}{\sqrt{n^3+1}}\right)_{n=1}^{\infty}$ decreases, and

$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+1}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\sqrt{n^3+1}}$ converges by the alternating series test.

So the series $\boxed{\text{converges conditionally}}$.

3. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(2x)^n}{n+1}$ converge absolutely?

Hint: Ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2x)^{n+1}}{(n+1)+1} \Big/ \frac{(2x)^n}{n+1}\right|$$

$$= \left|\frac{(2x)^{n+1}}{n+2} \cdot \frac{n+1}{(2x)^n}\right| = \left|\frac{(2x)^{n+1}}{(2x)^n} \cdot \frac{n+1}{n+2}\right| = \left|(2x)\left(\frac{n+1}{n+2}\right)\right|$$

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |2x| \cdot 1 = 2|x|$. So the series converges absolutely, by the ratio test, for $2|x| < 1$, or $\boxed{|x| < \frac{1}{2}}$.