MAT 137 **HW #5** Name SOLUTIONS

3/4/09 (due Monday 3/9/09)

10 points

Show work or otherwise justify your answers. Unsupported answers (i.e. calculator output) will not receive full credit. You may check your answers with a calculator or computer.

1.)(a) Determine all values of c for which the constant function $y(t) = c$ a solution of the equation

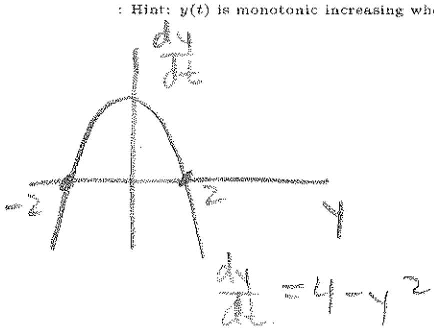
$$\frac{dy}{dt} = 4 - y^2.$$

$y = c \Rightarrow \frac{dy}{dt} = 0$, $4 - y^2 = 0$ iff $y = \pm 2$

$$\boxed{c = \pm 2}$$

(b) For what range of y-values are solutions monotonic increasing?

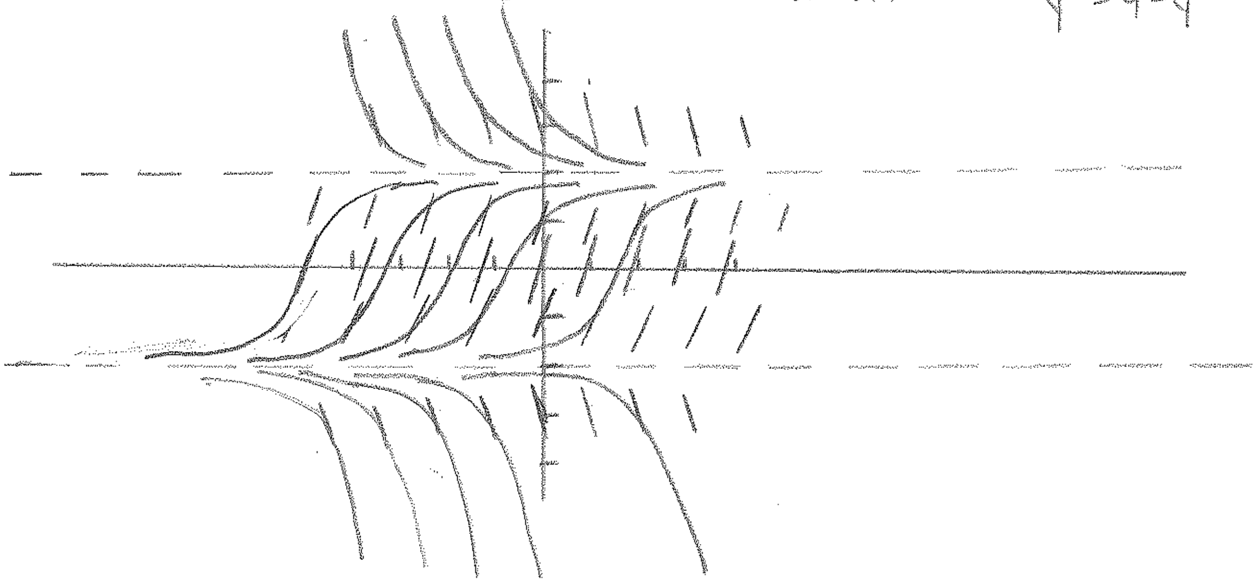
: Hint: $y(t)$ is monotonic increasing when $\frac{dy}{dt}$ is positive.



$\frac{dy}{dt} > 0$ for $\boxed{-2 < y < 2}$

(c) Sketch the direction field for this equation, and a few solutions.[1] Determine the limiting behavior of solutions as $t \to \infty$, for all possible initial values of $y_0 = y(0)$.

$y' = 4 - y^2$



[1] You can use JOde.

2. (a) Find the general solution $y(t)$ of the equation from Problem 1.

Hint: After some manipulation, integrate by using a partial fractions decomposition - be sure to factor the denominator first.

$$\frac{dy}{dt} = 4-y^2 \longrightarrow \frac{1}{4-y^2}\,dy = 1\,dt$$

$$\longrightarrow \int \frac{1}{4-y^2}\,dy = \int dt \longrightarrow \int\left(\frac{1/4}{2-y} + \frac{1/4}{2+y}\right)dy = t+C$$

$$\longrightarrow -\tfrac{1}{4}\ln|2-y| + \tfrac{1}{4}\ln|2+y| = t+C$$

$$\ln\left(\left|\frac{2+y}{2-y}\right|^{1/4}\right) = t+C$$

$$\left|\frac{2+y}{2-y}\right| = e^{4t}e^{4C} \longrightarrow \frac{2+y}{2-y} = \pm e^{4C}e^{4t} = ke^{4t}$$

$$2+y = (2-y)ke^{4t}$$

$$2+y = 2ke^{4t} - yke^{4t}$$

$$y + yke^{4t} = 2ke^{4t} - 2$$

$$y(1+ke^{t}) = 2ke^{t} - 2$$

$$\boxed{y = \frac{2ke^{4t}-2}{ke^{4t}+1}}$$

(b) Find the particular solution $y(t)$ with initial value $y(0) = 1$.

$$y(0) = 1 \longrightarrow 1 = \frac{2ke^{0}-2}{ke^{0}+1}$$

$$1 = \frac{2k-2}{k+1} \rightarrow k+1 = 2k-2 \longrightarrow k = 3$$

$$\boxed{y = \frac{6e^{4t}-2}{3e^{4t}+1}}$$

3. A linear first order ODE of the form $\frac{dy}{dt} = at + by$, where a and b are constants, always has a solution $y(t)$ which is a linear function. Find such a solution for the equation

$$\frac{dy}{dt} = 2t + 3y,$$

by writing $y(t) = mt + b$ and finding values for m and b so that $y(t)$ satisfies the equation[2].

$y(t) = mt + b \implies \frac{dy}{dt} = m$ and $2t + 3y = 2t + 3(mt+b)$

so y is a solution iff

$$m = 2t + 3(mt+b) \quad \text{for all } t$$

$$m = 2t + 3mt + 3b$$

$$m = (2+3m)t + 3b$$

For this to hold for all t, must have $2+3m = 0$ and $m = 3b$

$$\boxed{m = -\frac{2}{3},\ b = -\frac{2}{9}}$$

[2]You can check your answer by using JOde to plot solutions; you will see the linear solution as a major feature of the diagram.