MAT 137 **HW #4** Name SOLUTIONS

2/26/09 (due Monday 3/2/09)

10 points

Show work or otherwise justify your answers. Unsupported answers (i.e. calculator output) will not receive full credit. You may check your answers with a calculator or computer.

1. Determine whether or not the improper integral converges. If convergent, find the value.

(a) $\int_2^\infty \frac{1}{x^2-1}\,dx$

$\frac{1}{x^2-1} < \frac{1}{x^2}$ for $x \geq 2$, and $\int_2^\infty \frac{1}{x^2}\,dx$ converges $(p = 2 > 1)$, so

$\int_2^\infty \frac{1}{x^2-1}\,dx$ $\boxed{\text{converges.}}$

$$\int_2^\infty \frac{1}{x^2-1}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x^2-1}\,dx = \lim_{b\to\infty}\int_2^b\left(\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}\right)dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\ln(x-1) - \ln(x+1)\right]_2^b = \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)\Big|_2^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2}\ln\left(\frac{1}{3}\right)$$

$$= \frac{1}{2}\ln(1) - \frac{1}{2}\ln\left(\frac{1}{3}\right) = \boxed{\frac{1}{2}\ln(3)}$$

$(\doteq 0.5493)$

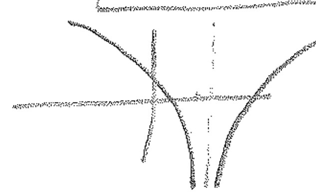
(b) $\int_0^1 \frac{1}{x^2-1}\,dx$

$$= \lim_{b\to 1^-}\int_0^b \frac{1}{x^2-1}\,dx$$

$$= \lim_{b\to 1^-} \frac{1}{2}\ln(|x-1|) - \frac{1}{2}\ln(x+1)\Big|_0^b$$

$$= \lim_{b\to 1^-} \frac{1}{2}\ln(|b-1|) - \lim_{b\to 1^-} \frac{1}{2}\ln(x+1) - \left[\frac{1}{2}\ln(1) - \frac{1}{2}\ln(1)\right]$$

$$= \lim_{b\to 1^-} \ln(|b-1|) - \frac{1}{2}\ln(2) \quad \boxed{\text{diverges}}$$



$y = \ln|x-1|$

2. Find the length of the curve $y = \ln(\cos(x))$, $0 \leq x \leq \frac{\pi}{3}$.

$$\frac{dy}{dx} = \frac{1}{\cos(x)} \cdot \sin(x) = \tan(x)$$

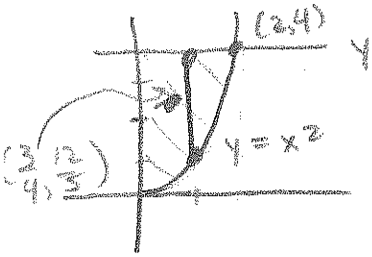
$$\text{Arclength} = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\,dx$$

$$= \int_0^{\pi/3} \sqrt{1 + \tan^2(x)}\,dx = \int_0^{\pi/3} \sqrt{\sec^2(x)}\,dx$$

$$= \int_0^{\pi/3} \sec(x)\,dx = \ln|\sec(x) + \tan(x)|\Big|_0^{\pi/3}$$

$$= \boxed{\ln(2 + \sqrt{3})} \quad (\doteq 1.31696)$$

3. (a) Find the centroid of the region R described by the system of inequalities $x^2 \le y \le 4,\ x \ge 0$.



area $A = \int_0^2 (4-x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_0^2 = \frac{16}{3}$

$$\bar{x} = \frac{\int_0^2 x(4-x^2)\,dx}{A} = \frac{\left[2x^2 - \frac{1}{4}x^4\right]_0^2}{16/3} = \boxed{\frac{3}{4}}$$

$$\bar{y} = \frac{\int_0^2 \frac{1}{2}\left[4^2 - (x^2)^2\right]dx}{A} = \frac{\frac{1}{2}\left[16x - \frac{1}{5}x^5\right]_0^2}{16/3} = \frac{12}{5}$$

$$\boxed{\text{centroid} = \left(\frac{3}{4}, \frac{12}{5}\right)} = (.75, 2.4)$$

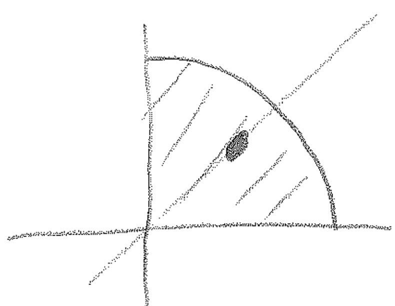
(b) Use your answer to (a), and Pappus' Theorem, to find the volume of the solids obtained by rotating the region R about (i) the y-axis, and (ii) the x-axis.

(i) About y-axis; Volume $= (2\pi\bar{x})(A) = 2\pi\left(\frac{3}{4}\right)\left(\frac{16}{3}\right)$

$\boxed{V = 8\pi}$ $(\doteq 25.13)$

(ii) About x-axis: Volume $= (2\pi\bar{y})(A)$
$= (2\pi)\left(\frac{12}{5}\right)\left(\frac{16}{3}\right)$ $\boxed{V = \frac{128\pi}{5}}$ $(\doteq 80.425)$

(c) Use Pappus' Theorem, and the formula for the volume of a sphere, to find the centroid of the quarter circle given by $x^2 + y^2 \le r^2,\ x \ge 0,\ y \ge 0$ without integrating.



By symmetry, $\bar{x} = \bar{y}$.

Volume of solid of revolution about y-axis $= \frac{1}{2} \times$ volume of sphere of radius r.

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$A = \frac{1}{4}(\pi r^2)$, and $V = 2\pi\bar{x}A$,

so $\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi r^3\right) = (2\pi\bar{x})\left(\frac{1}{4}\pi r^2\right)$

or $\frac{4}{3}\pi r = \pi^2\bar{x}$, so $\bar{x} = \frac{4r}{3\pi}$

$$\boxed{\text{centroid} = \left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)}$$