MAT 137 **HW #3** Name SOLUTIONS

2/6/09 (due Wednesday 2/11/09)

10 points

Show work or otherwise justify your answers. Unsupported answers (i.e. calculator output) will not receive full credit. You may check your answers with a calculator or computer.

1. Derive a reduction formula[1] for the integral $\int \sec^n(x)\,dx$, by first integrating by parts, then invoking a trigonometric identity, and then solving for the desired integral.

$$\int \sec^n(x)\,dx = \int \sec^{n-2}(x)\sec^2(x)\,dx = \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\,dx$$

$$\boxed{\begin{gathered} u=\sec^{n-2}(x),\ dv=\sec^2(x)dx \\ du=(n-2)\sec^{n-3}(x)\sec(x)\tan(x)dx \\ v=\tan(x)\end{gathered}}$$

$$= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\left(\sec^2(x)-1\right)dx$$

So $\int \sec^n(x)\,dx = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx - (n-2)\int \sec^n(x)\,dx$

Solving for $\int \sec^n(x)\,dx$ yields

$$\boxed{\int \sec^n(x)\,dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx}$$

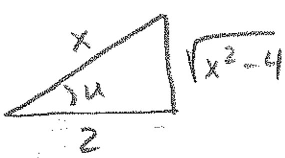
2. Find the integral $\int \frac{1}{x^2-4}\,dx$ by two different methods:

(a) using a trigonometric substitution.

$x = 2\sec(u)$

$dx = 2\sec(u)\tan(u)\,du$

$x^2-4 = 4\sec^2(u) - 4 = 4\tan^2(u)$



$\sec(u) = \frac{x}{2} \Longrightarrow$

$\csc(u) = \frac{x}{\sqrt{x^2-4}}$

$\cot(u) = 2/\sqrt{x^2-4}$

$$= \int \frac{2\sec(u)\tan(u)}{4\tan^2(u)}\,du = \frac{1}{2}\int \frac{\sec(u)}{\tan(u)}\,du = \frac{1}{2}\int \frac{1}{\sin(u)}\,du$$

$$= \frac{1}{2}\int \csc(u)\,du$$

$$= -\frac{1}{2}\ln\left|\csc(u)+\cot(u)\right| + C$$

$$= \boxed{-\frac{1}{2}\ln\left|\frac{x}{\sqrt{x^2-4}} + \frac{2}{\sqrt{x^2-4}}\right| + C}$$

$$\left(= -\frac{1}{2}\ln\left|\frac{x+2}{\sqrt{x^2-4}}\right| + C = -\frac{1}{2}\ln\left|\frac{\sqrt{x+2}}{\sqrt{x-2}}\right| + C = \frac{1}{4}\ln\left(\frac{x-2}{x+2}\right) + C\right)$$

(b) using a partial fractions decomposition.

$$\frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\Longrightarrow 1 = A(x-2) + B(x+2)$$

$x=2: \quad 1 = 4B, \quad B = 1/4$

$x=-2: \quad 1 = -4A, \quad A = -1/4$

$$\frac{1}{x^2-4} = \frac{-1/4}{x+2} + \frac{1/4}{x-2} \quad \text{so} \quad \int \frac{1}{x^2-4}\,dx = \int \frac{-1/4}{x+2}\,dx + \int \frac{1/4}{x-2}\,dx$$

$$= \boxed{-\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + C}$$

[1] That is, find a formula that expresses the integral in terms of an integral of a smaller power of secant.

3. Find these integrals.

(a) $\int \arcsin(x)\,dx$ $= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$

integrate by parts

$u = \arcsin(x)$

$dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

$v = x$

$z = 1-x^2$

$dz = -2x\,dx$

$= x\arcsin(x) - \left(-\tfrac{1}{2}\right)\int z^{-1/2}\,dz$

$= x\arcsin(x) + \left(\tfrac{1}{2}\right)\left(\frac{z^{1/2}}{1/2}\right) + C$

$= x\arcsin(x) + z^{1/2} + C$

$= \boxed{x\arcsin(x) + \sqrt{1-x^2} + C}$

(b) $\int \frac{1}{1+\sqrt[3]{x}}\,dx$

$u = 1+\sqrt[3]{x}$

$u-1 = \sqrt[3]{x}$

$(u-1)^3 = x \longrightarrow dx = 3(u-1)^2\,du$

$\int \frac{1}{1+\sqrt[3]{x}}\,dx = \int \frac{3(u-1)^2}{u}\,du$

$= 3\int u - 2 + \frac{1}{u}\,du$

$= 3\left[\tfrac{1}{2}u^2 - 2u + \ln|u|\right] + C$

$= \boxed{3\left[\tfrac{1}{2}(1+\sqrt[3]{x})^2 - 2(1+\sqrt[3]{x}) + \ln|1+\sqrt[3]{x}|\right] + C}$

(c) $\int \frac{27x-18}{(x-1)(x^2+x-2)}\,dx$

$\frac{27x-18}{(x-1)(x^2+x-2)} = \frac{27x-18}{(x-1)(x-1)(x+2)}$

$= \frac{27x-18}{(x-1)^2(x+2)}$

$\frac{27x-18}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

$27x-18 = A(x+2) + B(x-1)(x+2) + C(x-1)^2$

$x=1$: $27\cdot 1 - 18 = A(3) + 0 + 0 \Longrightarrow 9 = 3A$, $\boxed{A=3}$

$x=-2$: $(27)(-2) - 18 = 0 + 0 + C(-3)^2 \Longrightarrow -72 = 9C$ $\boxed{C=-8}$

$x=0$: $0 - 18 = A(2) + B(-2) + C(-1)^2$

or $-18 = 2A - 2B + C \Longrightarrow \boxed{B=8}$

So $\int \frac{27x-18}{(x-1)(x^2+x-2)}\,dx = \int\left(\frac{3}{(x-1)^2} + \frac{8}{(x-1)} - \frac{8}{(x+2)}\right)dx = \boxed{-\frac{3}{x-1} + 8\ln|x-1| - 8\ln|x+2| + C}$