MAT 137-7 **Final Exam** Name SOLUTIONS

5/6/09

110 points

Calculators may be used, but you must show all work - unsupported answers (e.g., calculator output) will receive minimal credit

1.(12) Evaluate these indefinite and definite integrals. You must show all work, step-by-step, to receive credit - no calculator answers.

(a) $\int_0^\pi \sin^3(x)\,dx = \int_0^\pi \sin^2(x)\sin(x)\,dx = \int_0^\pi (1-\cos^2(x))\sin(x)\,dx$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$= -\int_1^{-1} (1-u^2)\,du$

$= -\left(u - \frac{u^3}{3}\right)\Big|_1^{-1} = \boxed{\frac{4}{3}}$

(b) $\int x^2 \sin(x)\,dx$

$u = x^2 \quad dv = \sin(x)dx$

$du = 2x\,dx \quad v = -\cos(x)$

$= -x^2\cos(x) + \int 2x\cos(x)dx \qquad u = 2x \quad dv = \cos(x)dx$

$du = 2dx \quad v = \sin(x)$

$= -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\,dx$

$= \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C}$

(c) $\int \ln(x)\,dx$

$u = \ln(x) \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$= x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int 1\,dx$

$= \boxed{x\ln(x) - x + C}$

2.(5) (a) Show that, by change of variables, $\int_0^4 e^{\sqrt{x}}\,dx$ is equal to $\int_0^2 2ue^u\,du$.

$u = \sqrt{x} \longrightarrow x = u^2$

$x = 0 \Rightarrow u = 0 \quad dx = 2u\,du$

$x = 4 \Rightarrow u = 2$

$= \int_0^2 2ue^u\,du$

(b) Evaluate the second integral in part (a).

$$\int_0^2 2ue^u\,du = 2ue^u\Big|_0^2 - \int_0^2 2e^u\,du$$

$z = 2u \quad dw = e^u du$

$dz = 2du \quad w = e^u$

$$= 4e^2 - 2e^u\Big|_0^2$$

$$= \boxed{2e^2 + 2}$$

3.(10) (a) Find the general solution of the ODE $\frac{dy}{dx} = 3y$.

$$\frac{1}{y}dy = 3dx$$

$$\int \frac{1}{y}dy = \int 3\,dx$$

$$\ln|y| = 3x + C$$

$$|y| = e^{3x+C} = e^C e^{3x}$$

$$y = \pm e^C e^{3x} = ke^{3x}$$

$k \neq 0$

$y = 0$ also satisfies the equation

$$\boxed{y = ke^{3x}}$$

(b) Solve the initial value problem $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = 2$.

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$x = 0 \Rightarrow y = 2 \quad \boxed{2 = C}$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + 2$$

$$y^2 = x^2 + 4$$

$$\boxed{y = +\sqrt{x^2+4}}$$

(so that $y = +2$ when $x = 0$)

4.(8) A 4% salt solution is pumped into a tank at 2 liters per second, and the mixture drains from the tank at the same rate. Assume the tank initially contains 50 liters of 2% salt solution. Set up an initial value problem (a first order ODE with initial condition) for the amount Q of salt in solution as a function of time t. *Do not solve.*

$$\frac{dQ}{dt} = (2)(0.04) - (2)\left(\frac{Q}{50}\right), \quad Q(0) = (0.02)(50)$$

or

$$\boxed{\frac{dQ}{dt} = 0.08 - 0.04Q, \quad Q(0) = 1}$$

5.(8) Consider the autonomous first order ODE $\dfrac{dy}{dx} = 4y - y^2$.

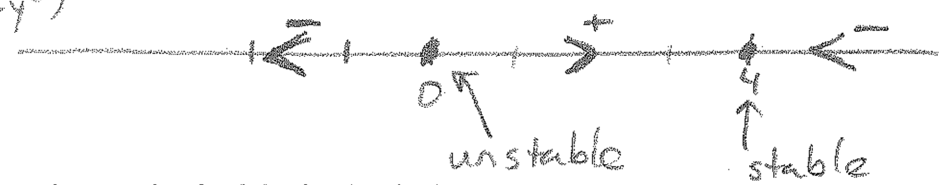
(a) Find all equilibrium solutions, that is, solutions of the form $y = c$ for some constant c.

$$y = c \Rightarrow \frac{dy}{dx} = 0 \Rightarrow 4c - c^2 = 0, \quad c(4-c) = 0, \quad c = 0, \ c = 4$$

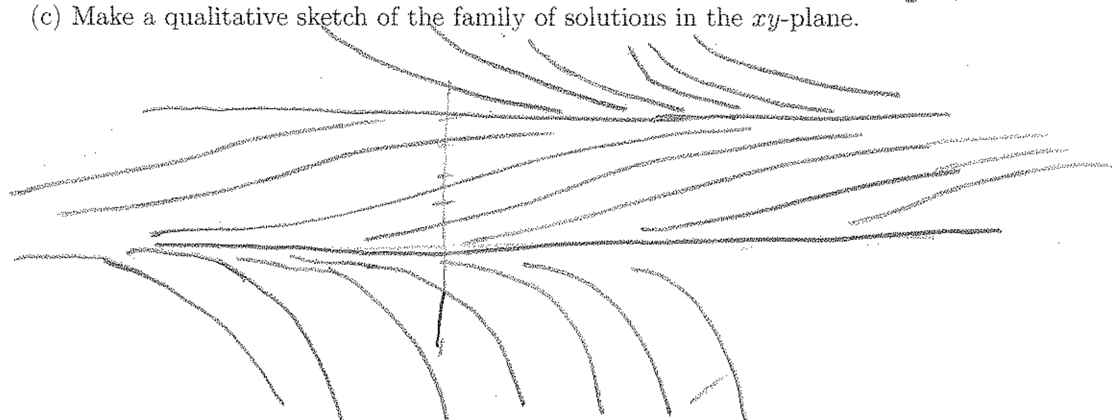
$$\boxed{y = 0 \text{ and } y = 4}$$

(b) Sketch the "phase line" (the y-axis), indicating with arrows in which intervals solutions are increasing or decreasing. Determine the stability of each equilibrium point.

$\operatorname{sign}\left(\frac{dy}{dx}\right) = \operatorname{sign}(4y - y^2)$



(c) Make a qualitative sketch of the family of solutions in the xy-plane.



6.(5) Use the ratio test to show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ diverges.

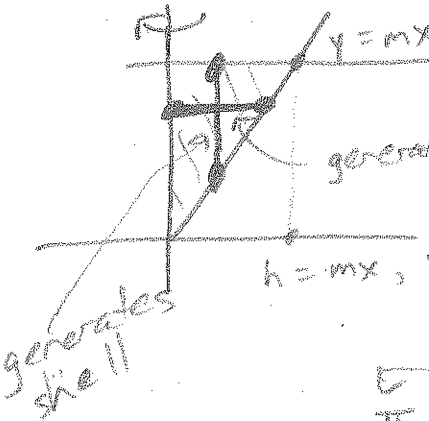
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} = \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \frac{n+1}{1} \cdot \frac{1}{(2n+2)(2n+1)}$$

$$= \frac{1}{2(2n+1)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2(2n+1)} = 0.$$ Since the limit is less than one, the series converges.

7.(15) (a) Set up *two* different integrals for the volume of a cone of radius a and height h, one using slices (the disk method), and the other using cylindrical shells. Note that such at cone is generated by rotating the region bounded by the lines $y = mx$, $y = h$, and $x = 0$ about the y-axis, where the slope m is equal to $\frac{h}{a}$. Evaluate one of the integrals to obtain the formula $V = \frac{1}{3}\pi a^2 h$.



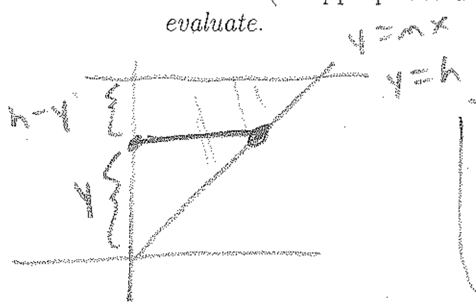
$h = mx$, $x = \frac{h}{m} = a$

$$\boxed{V = \int_0^h \pi \left(\frac{y}{m}\right)^2 dy, \quad m = \frac{h}{a}}$$

or $\boxed{V = \int_0^a 2\pi x (h - mx)\, dx}$

$$V = \frac{\pi}{m^2}\int_0^h x^2\, dx = \frac{\pi}{m^2}\cdot\frac{1}{3}h^3 = \frac{1}{3}\pi\left(\frac{h}{m}\right)^2 h = \boxed{\tfrac{1}{3}\pi a^2 h}$$

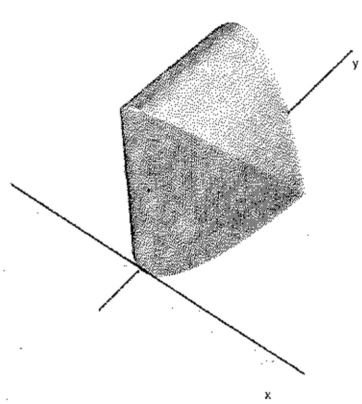
(b) Suppose the mass density of the cone at each point is equal to the square of the distance to the base (in appropriate units). Set up an integral for the total mass of the cone. *Do not evaluate.*

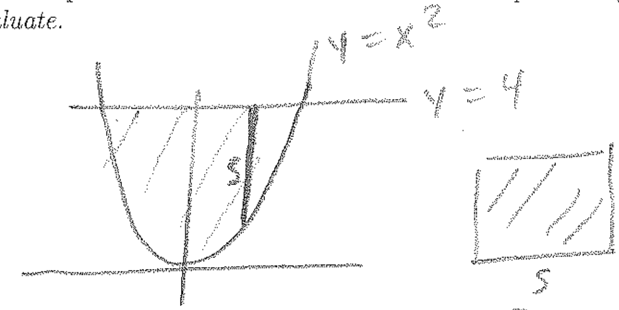


density at height y = $(h-y)^2$

$$\boxed{\text{mass} = \int_0^h \pi\left(\frac{y}{m}\right)^2 (h-y)^2\, dy}$$

(c) The (horizontal) base of the solid is the region bounded by $y = x^2$ and $y = 4$. Vertical cross sections *perpendicular* to the x-axis are *squares* with one side in the base. Set up an integral for the volume of the solid. *Do not evaluate.*





$$A(x) = s^2 = (4 - x^2)^2$$

$$\boxed{V = \int_{-2}^{2} (4-x^2)^2\, dx}$$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$

$|a_n| = \frac{1}{\sqrt{n^2+1}}$. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges by the limit comparison test, comparing with the divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \sqrt{\lim_{n\to\infty} \frac{n^2}{n^2+1}} = \sqrt{1} = 1 \neq 0$$

But $a_n = \frac{1}{\sqrt{n^2+1}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges by the alternating series test. The series converges conditionally.

10.(12) (a) Find the sum of the power series $\sum_{n=1}^{\infty} x^{2n}$, and determine the interval of convergence. (Note: For fixed x the series is geometric.)

$= x^2 + x^4 + x^6 + x^8 + \cdots$

geometric w/ ratio x^2, first term x^2, so $\sum_{n=1}^{\infty} x^{2n}$ converges to $\boxed{\frac{x^2}{1-x^2}}$ for $|x^2| < 1$, or $\boxed{|x| < 1}$

(b) Show that the power series $\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$ converges for all values of x.

absolute ratio test

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2 |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{n^2 |x|^n}$$

$$= \left(\frac{n+1}{n}\right)^2 \cdot \left(\frac{n!}{(n+1)!}\right) |x|$$

$$= \left(\frac{n+1}{n}\right)^2 \left(\frac{1}{n+1}\right) |x|$$

So $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \left(\frac{1}{n+1}\right) |x| = 1^2 \cdot 0 \cdot |x| = 0$ for any x.

since $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ the series converges absolutely for all x.

(c) Find the third degree Taylor polynomial (centered at $a = 0$) for the function $f(x) = (1+x)^{\frac{5}{2}}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$(1+x)^{5/2}$	1	1
1	$5/2\,(1+x)^{3/2}$	$5/2$	$5/2$
2	$15/4\,(1+x)^{1/2}$	$15/4$	$15/8$
3	$\frac{15}{8}(1+x)^{-1/2}$	$15/8$	$5/16$

$$P_3(x,0) = \boxed{1 + \frac{5}{2}x + \frac{15}{8}x^2 + \frac{5}{16}x^3}$$

8.(10) (a) Use a partial fractions decomposition to evaluate the indefinite integral $\int \frac{1}{x^2+x}\,dx$

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \Longrightarrow 1 = A(x+1) + Bx$$

$$\Longrightarrow (x=0) \quad 1 = A \text{ and } (x=-1), \quad -1 = B$$

$$\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln(x) - \ln(x+1) + C$$

(b) Show that the improper integral $\int_1^\infty \frac{1}{x^2+x}\,dx$ converges, and find its value.

$$\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+x}\,dx =$$

$$\lim_{b\to\infty} [\ln(b) - \ln(b+1)] - [\ln(1) - \ln(2)] = \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) + \ln(2)$$

$$= \ln\left(\lim_{b\to\infty} \frac{b}{b+1}\right) + \ln(2) =$$

$$\ln(1) + \ln(2)$$

$$\boxed{= \ln(2)}$$

(c) Give two different proofs that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges, using

(i) the integral test, and

$\frac{1}{x^2+x}$ is decreasing and positive for $1 \le x < \infty$, and $\int_1^\infty \frac{1}{x^2+x}\,dx$ converges by part (b), therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by the integral test.

(ii) the basic comparison test, comparing with a p-series.

$0 < \frac{1}{n^2+n} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges by comparison.
$(p=2)$

9.(10) Determine whether the given infinite series converges absolutely, conditionally, or diverges. Justify your answers with convergence tests.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^3+1}$ $\qquad |a_n| = \frac{\ln(n)}{n^3+1}$

For n sufficiently large, $\frac{\ln(n)}{n^3+1} < \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3+1}$ converges by comparison. Then

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^3+1} \text{ converges absolutely.}}$

11.(15) Let $\mathbf{v} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$

(a) Find $3\mathbf{v} - 2\mathbf{w}$.

$$3(4,1,2) - 2(1,-1,5) = (12,3,6) - (2,-2,10)$$
$$= (10, 5, -4) = \boxed{10\hat{i} + 5\hat{j} - 4\hat{k}}$$

(b) Find the magnitude $\|\mathbf{v}\|$ of $\mathbf{v}$, and then find a *unit vector* $\hat{\mathbf{u}}$ that points in the same direction as $\mathbf{v}$.

$$\|\vec{v}\| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$$
$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{21}}(4,1,2) = \left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right) = \boxed{\frac{4}{\sqrt{21}}\hat{i} + \frac{1}{\sqrt{21}}\hat{j} + \frac{2}{\sqrt{21}}}$$

(c) Determine whether or not $\mathbf{v}$ and $\mathbf{w}$ are orthogonal (i.e., perpendicular). If not, determine the angle between $\mathbf{v}$ and $\mathbf{w}$.

$$\vec{v} \cdot \vec{w} = (4,1,2) \cdot (1,-1,5)$$
$$= (4)(1) + (1)(-1) + (2)(5) = 13$$

$\neq 0$ so not perpendicular. $\cos(\theta) = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \, \|\vec{w}\|} = \dfrac{13}{\sqrt{21}\sqrt{27}} \Rightarrow$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{567}}\right) \doteq$$

(d) Find the cross product $\mathbf{v} \times \mathbf{w}$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix} = (1 \cdot 5 - 2 \cdot (-1), -(4 \cdot 5 - 2 \cdot 1), 4 \cdot (-1) - 1 \cdot 1)$$
$$= (7, -18, -5) = \boxed{7\hat{i} - 18\hat{j} - 5\hat{k}}$$

(e) Find the scalar and vector projections of $\mathbf{v}$ on $\mathbf{w}$.

$$\text{scalar projection} = \vec{v} \cdot \left(\frac{\vec{w}}{\|\vec{w}\|}\right) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|} = \boxed{\frac{13}{\sqrt{27}}}$$

$$\text{vector projection} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w} = \left(\frac{13}{27}\right)(1,-1,5$$
$$= \left(\frac{13}{27}, \frac{-13}{27}, \frac{65}{27}\right)$$
$$= \boxed{\frac{13}{27}\hat{i} - \frac{13}{27}\hat{j} + \frac{65}{27}\hat{k}}$$