MAT 137-7 **Exam 3** Name SOLUTIONS

04/17/09

70 points

Calculators may be used, but you must show all work - unsupported answers (e.g., calculator output) will receive minimal credit

1.(12) Consider the infinite series $\sum_{n=1}^{\infty} \frac{n+1}{n}$.

(a) Is the sequence of *terms* (i) positive? (ii) increasing? (iii) convergent? Justify your answers.

$(a_n)_{n=1}^{\infty} = \left(\frac{n+1}{n}\right)_{n=1}^{\infty} = \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots\right)$

is positive, decreasing, and converges to 1.

(b) Is the sequence of *partial sums* (i) increasing? (ii) bounded above? (iii) convergent? If the sequence of partial sums is convergent, find the limit. Justify your answers, and explain how they are related to the answers in part (a).

$(s_n)_{n=1}^{\infty} = \left(\frac{2}{1}, \frac{2}{1}+\frac{3}{2}, \frac{2}{1}+\frac{3}{2}+\frac{4}{3}, \dots\right)$

$= (2, 3\frac{1}{2}, 4\frac{5}{6}, \dots)$ is increasing (because the terms are positive), not bounded above (because the terms don't converge to zero), and hence not convergent.

(c) Is the infinite series convergent? Explain your answer.

The infinite series is not convergent because the sequence of partial sums is not convergent.

(d) Is the infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n}$ convergent? Justify your answer.

This series diverges by the n^{th} term (or divergence) test, since $\lim_{n\to\infty} (-1)^{n-1}\left(\frac{n+1}{n}\right)$ doesn't exist (because the signs alternate and the magnitudes converge to 1, not 0.

2.(6) Find $\lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{4^n}\right)$, if the limit exists.

$= \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$ geometric, first term $\frac{1}{4}$, common ratio $1/4$, which is between -1 and 1, so the series converges to $\frac{1/4}{1 - 1/4} = \boxed{1/3}$

3.(20) Write out the first few terms of each series, and determine whether the series converges. Justify your answers.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

The series diverges. Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$ using the limit comparison test: $\lim_{n\to\infty} \frac{1/n}{1/(2n+1)} = \lim_{n\to\infty} \frac{2n+1}{n} = \lim_{n\to\infty} \left(2 + \frac{1}{n}\right)$

$= 2 + 0 = 2 \neq 0, \infty$; $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series. ($p = 1 \geq 1$). Alternatively: $\frac{1}{2n+1} > \frac{1}{2n+n} = \frac{1}{3n}$ and $\sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n^3+1}} = \frac{1}{1+\sqrt{2}} + \frac{1}{2+\sqrt{9}} + \frac{1}{3+\sqrt{28}} + \cdots$

The series converges. Compare with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which is a convergent p-series ($p = 3/2 > 1$). $0 < \frac{1}{n+\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n+12)!} = \frac{1}{13!} + \frac{4}{14!} + \frac{9}{15!} + \cdots$

The series converges by the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{((n+1)+12)!} \Big/ \frac{n^2}{(n+12)!} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+13)!} \cdot \frac{(n+12)!}{n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{(n+12)!}{(n+13)!}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{n+13}$$

$$= (1+0)^2 \cdot 0 = 0$$

Since $0 < 1$ the series converges.

(d) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)} = \frac{1}{\ln(2)} - \frac{1}{\ln(3)} + \cdots$

The series converges by the alternating series test.

$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ and

$\left(\frac{1}{\ln(n)}\right)_{n=2}^{\infty}$ is decreasing. Therefore the alternating series $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ converges.

4.(10) Determine whether these series converge absolutely, converge conditionally, or diverge.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$ $\boxed{\text{Converges absolutely}}$, by the ratio test

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n!}\right| = \sum_{n=2}^{\infty} \frac{1}{n!} \text{ converges: } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ (Compare with Problem 3(d).)

$\boxed{\text{Converges conditionally}}$: $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$ converges by 3(d), but $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$: $\frac{1}{\ln(n)} > \frac{1}{n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series. ($p = 1 \geq 1$.)

(c) $\sum_{n=2}^{\infty} \frac{\sin(n)}{n!}$

$\boxed{\text{converges absolutely}}$, by comparison and ratio tests: $\left|\frac{\sin(n)}{n!}\right| \leq \frac{|\sin(n)|}{n!} \leq \frac{1}{n!}$, and $\sum_{n=2}^{\infty} \frac{1}{n!}$ converges by 4(a).

(d) $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+\frac{1}{n}}$

$\boxed{\text{diverges}}$: $\lim_{n\to\infty} \frac{(-1)^n}{1+1/n}$ doesn't exist, because the sequence alternates and $\lim_{n\to\infty} \frac{1}{1+1/n} = 1 \neq 0$.

5.(8) For what values of x does the power series $\sum_{n=1}^{\infty} \frac{n^3 x^n}{5^n}$ converge?

absolute ratio test $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^3 |x|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n^3 |x|^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \left(\frac{1}{5}\right) |x|$

$= 1 \cdot \frac{1}{5} |x| < 1$ iff $|x| < 5$

radius of convergence is 5.

$x = 5$: $\sum_{n=1}^{\infty} \frac{n^3 5^n}{5^n} = \sum_{n=1}^{\infty} n^3$ diverges. ($\lim_{n\to\infty} n^3$ doesn't exist.)

$x = -5$: $\sum_{n=1}^{\infty} \frac{n^3(-5)^n}{5^n}$

$= \sum_{n=1}^{\infty} (-1)^n n^3$ diverges

($\lim_{n\to\infty} (-1)^n n^3$ doesn't exist.)

So the series converges for $\boxed{-5 < x < 5}$

6.(8) (a) Show that, for fixed x, the series $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ is geometric, and find the common ratio.

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots$$

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(x-1)^{n+1}}{(-1)^n (x-1)^n} = -(x-1) \text{ constant}$$

so series is geometric with ratio $-(x-1)$.

(b) Find the sum of the series in (a). For what values of x does the series converge?

$$\sum_{n=0}^{\infty}(-1)^n(x-1)^n = \frac{1}{1-[-(x-1)]} = \frac{1}{1+(x-1)} = \boxed{\frac{1}{x}}$$

for $|x-1|<1$

(c) Use (b) to find a power series centered at 1 (i.e., in powers of $(x-1)$, as in (a)) that converges to $\ln(x)$ for $0 < x < 2$.

Hint: Integrate term-by-term.

$$\int \frac{1}{x}\,dx = \ln(x) = \int \left(\sum_{n=0}^{\infty}(-1)^n(x-1)^n\right)dx$$

$$= \sum_{n=0}^{\infty}\int(-1)^n(x-1)^n\,dx = \boxed{\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(x-1)^{n+1}}{n+1}}$$

Note: Both $\ln(x)$ and $\sum(-1)^n\frac{(x-1)^{n+1}}{n+1}$ are zero when $x=1$, so no constant term is required.

$|x-1|<1 \iff 0<x<2$

7.(6) Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges to e^{2x} for $|x|<1$. Show $c_0 = 1$, and then find c_1, c_2, and c_3 by repeatedly differentiating the series term-by-term and evaluating at $x=0$..

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
$$= e^{2x}$$

At $x=0$, sum $= c_0$ and $e^{2x} = 1$, so $\boxed{c_0 = 1}$

Differentiating: $c_1 + 2c_2 x + 3c_3 x^2 + \cdots$
$$= 2e^{2x}$$

$$2c_2 + 6c_3 x + \cdots = 4e^{2x}$$
$$6c_3 + \cdots = 8e^{2x}$$

at $x=0$, obtain: $\boxed{c_1 = 2}$, $2c_2 = 4$ so $\boxed{c_2 = 2}$

and $6c_3 = 8$ so $\boxed{c_3 = \frac{4}{3}}$