MAT 137-7 **Exam 2** Name SOLUTIONS

03/13/09

65 points

Calculators may be used, but you must show all work - unsupported answers (e.g., calculator output) will receive minimal credit

1.(10) Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2}{y(1+x^3)}$, $y(0) = 2$.

$$y\,dy = \frac{x^2}{1+x^3}\,dx \longrightarrow \int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

$$\tfrac{1}{2}y^2 = \tfrac{1}{3}\ln(1+x^3) + C, \quad y(0) = 2 \implies$$

$$\tfrac{1}{2}2^2 = \tfrac{1}{3}\ln(1+0) + C \implies 2 = C$$

$$\tfrac{1}{2}y^2 = \tfrac{1}{3}\ln(1+x^3) + 2 \quad \text{or} \quad \boxed{y = \sqrt{\tfrac{2}{3}\ln(1+x^3)+4}}$$

2.(9) Consider the autonomous first order ODE $\dfrac{dy}{dx} = 4y - y^3$.

(a) Find all equilibrium solutions, that is, solutions of the form $y = c$ for some constant c.

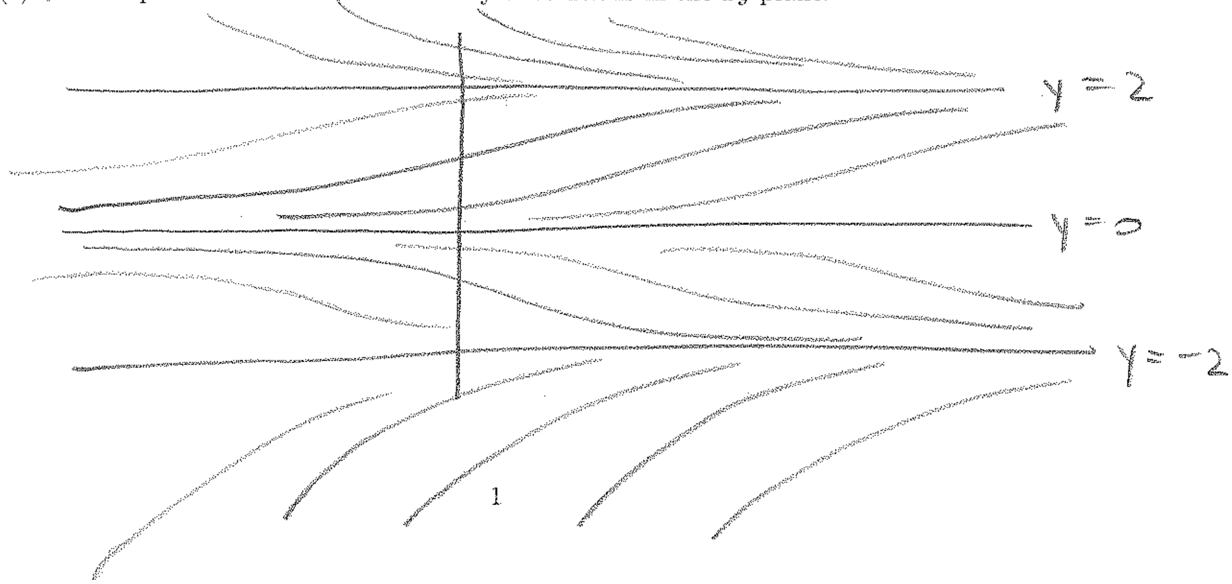
$y = c \implies \dfrac{dy}{dx} = 0$ so satisfies ODE iff $4c - c^3 = 0$

$c(4-c^2) = 0$, $c = 0, +2, -2$. Equilibrium solutions are

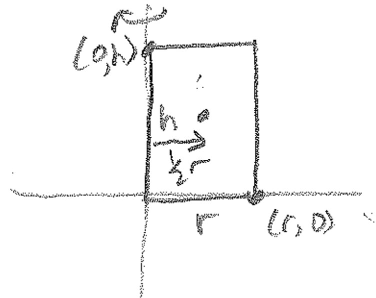
$\boxed{y = 0,\ y = 2,\ \text{and } y = -2}$

(b) Sketch the "phase line" (the y-axis), indicating with arrows in which intervals solutions are increasing or decreasing. Determine the stability of each equilibrium point.

(c) Make a qualitative sketch of the family of solutions in the xy-plane.



3.(8) Without integrating, verify that Pappus' Theorem gives the correct formula for the volume of a cylinder of radius r and height h, as the solid obtained by rotating a rectangle about one of its sides. (Use symmetry to locate the centroid of the rectangle.)



centroid is located at $(\frac{1}{2}r, \frac{1}{2}h)$

area is rh, so by Pappus' Thm,

$$\text{Volume} = (2\pi)(\tfrac{1}{2}r)(rh) = \pi r^2 h \checkmark$$

4.(12) (a) Without integrating, prove the that integral $\int_1^\infty \frac{1}{x^2+x}\,dx$ converges.

$0 < \frac{1}{x^2+x} \leq \frac{1}{x^2}$ for all $x \geq 1$, and $\int_1^\infty \frac{1}{x^2}\,dx$ converges, so $\int_1^\infty \frac{1}{x^2+x}\,dx$ converges by the comparison thm.

(b) Find the value of the integral in part (a).

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1} \text{ so } \int \frac{1}{x^2+x}\,dx$$

$$= \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln(x) - \ln(x+1) = \ln\left(\frac{x}{x+1}\right).$$

Then $\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+x}\,dx = \lim_{b\to\infty} \ln\left(\frac{b}{b+1}\right) - \ln\left(\frac{1}{2}\right)$

$= \ln(1) - \ln(\frac{1}{2})$

$= \boxed{\ln(2)}$

(c) Using the comparison theorem for improper integrals, explain why the following statement is true: if $f(x) > 0$ for $x \geq 1$ and $\int_1^\infty f(x)\,dx$ converges, then $\int_1^\infty (f(x))^2\,dx$ converges.

Hint: What can you say about $f(x)$, for large x, based on the assumption that $\int_1^\infty f(x)\,dx$ converges?

If $\int_1^\infty f(x)\,dx$ converges, then $\lim_{x\to\infty} f(x) = 0$, so $f(x) < 1$ for all x sufficiently large. Then $f(x)^2 < f(x)$ for all x sufficiently large. Since $0 < (f(x))^2$ and $\int_1^\infty f(x)$ converges, $\int_1^\infty (f(x))^2\,dx$ converges by comparison.

(d) Determine which of these improper integrals converge and which diverge. (No written justification is required for full credit.)

(i) $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$ (ii) $\int_1^3 \frac{1}{x-1}\,dx = \int_0^2 \frac{1}{u}\,du$

$u = 3-x$
$du = -dx$
$x = 0 \to u = 3$
$x = 3 \to u = 0$

$= -\int_3^0 \frac{1}{\sqrt{u}}\,du$

$= \int_0^3 \frac{1}{\sqrt{u}}\,du$ $\boxed{\text{converges}}$

$u = x-1$
$du = dx$
$x = 1 \to u = 0$
$x = 3 \to u = 2$

$\boxed{\text{diverges}}$

5.(8) Radioactive effluent is flowing into a pond at the rate of 1000 liters per day. The concentration of pollutant in the effluent is given by $c(t) = 100e^{-.3t}$ grams per liter. Assume the solution in the pond is uniformly mixed, and flows out of the pond at 1000 liters per day, so that the volume of solution remains constant. Assume the volume of the pond is 10^6 liters and the initial concentration of pollutant in the pond is zero.

Write a first order ODE, with initial value, for the amount $Q(t)$ of pollutant in the pond (in grams) as a function of time (in days). *Do not solve.*

$$\frac{dQ}{dt} = \underbrace{(1000)}_{\text{liters/day}}\underbrace{(100e^{-0.3t})}_{\text{grams/liter}} - \underbrace{(1000)}_{\text{liters/day}}\underbrace{\left(\frac{Q}{10^6}\right)}_{\text{grams/liter}}$$

or $\boxed{Q' = 10^5 e^{-0.3t} - 10^{-3}Q \text{ and } Q(0) = 0.}$

6.(10) For each differential equation below, identify which picture on the next page best illustrates the solution curves. *Do not solve the equations.*

(a) $\frac{dy}{dx} = y^2 - 4$ a

$y = \pm 2$ equilibrium solutions, $y' < 0$ for $-2 < y < 2$

(b) $\frac{dy}{dx} = 4 - y^2$ e

$y = \pm 2$ equilibrium solutions
$y' > 0$ for $-2 < y < 2$

(c) $\frac{dy}{dx} = y - 2$

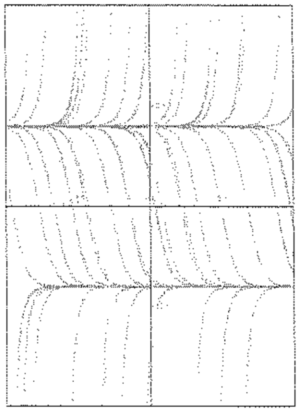
$y = 2$ equilibrium solution

f

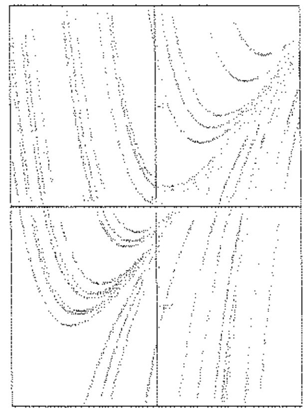
(d) $\frac{dy}{dx} = x - y$

$y = mx + b$ is a solution iff
$m = x - (mx + b)$
$m = (1-m)x - b$
so m must $= 1$ and
$b = -1$

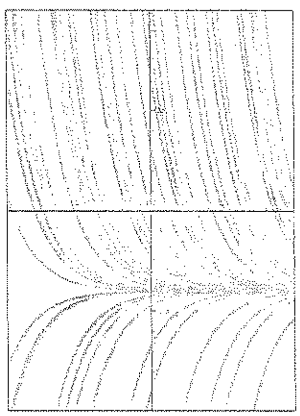
b



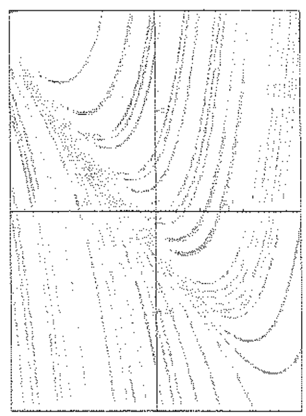
(a)



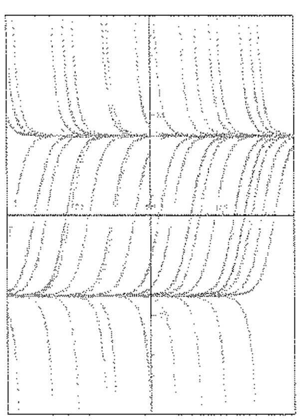
(b)



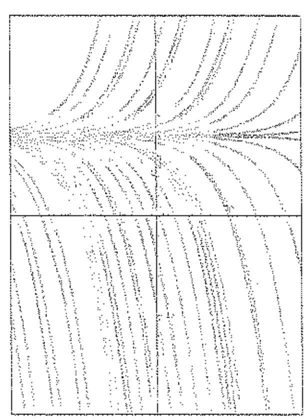
(c)



(d)



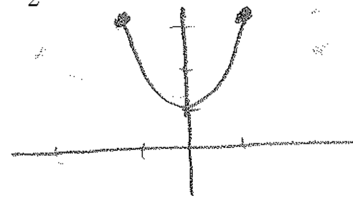
(e)



(f)

7.(5) Set up an integral for the length of the curve given by $y = \dfrac{e^x + e^{-x}}{2}$, $-1 \le x \le 1$. *Do not evaluate.*

Note: This curve has the shape of a cable hanging between two posts of equal height.



$$\frac{dy}{dx} = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\text{Length} = \int_{-1}^{1} \sqrt{1 + \left[\tfrac{1}{2}(e^x - e^{-x})\right]^2}\, dx \quad \longleftarrow \text{correct answer}$$

$$\Big(= \int_{-1}^{1} \sqrt{1 + \tfrac{1}{4}e^{2x} - \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}}\, dx$$

$$= \int_{-1}^{1} \sqrt{\tfrac{1}{4}e^{2x} + \tfrac{1}{2} + \tfrac{1}{4}e^{-2x}}\, dx = \int_{-1}^{1} \sqrt{\left(\tfrac{1}{2}(e^x + e^{-x})\right)^2}\, dx$$

$$= \int_{-1}^{1} \tfrac{1}{2}(e^x + e^{-x})\, dx$$

$$= e - \frac{1}{e} \Big)$$

8.(5) Show that the second order ODE $y'' + 4y' + 4y = 0$ has a solution of the form $y = e^{kx}$ for some constant k, and find the value of k.

$$y = e^{kx} \implies y' = ke^{kx}$$

$$\text{and } y'' = k^2 e^{kx}$$

$$y'' + 4y' + 4y = 0 \text{ iff}$$

$$k^2 e^{kx} + 4ke^{kx} + 4e^{kx} = 0 \text{ for all } x.$$

$$(k^2 + 4k + 4)e^{kx} = 0 \text{ for all } x.$$

$$e^{kx} \ne 0 \text{ for all } x, \text{ so}$$

$$y'' + 4y' + 4y = 0 \text{ iff } k^2 + 4k + 4 = 0$$

$$(k+2)^2 = 0$$

$$\boxed{k = -2}$$

$y = e^{-2x}$ is a solution,

and is the only solution of the form $y = e^{kx}$.