MAT 137-7 **Exam 1** Name SOLUTIONS

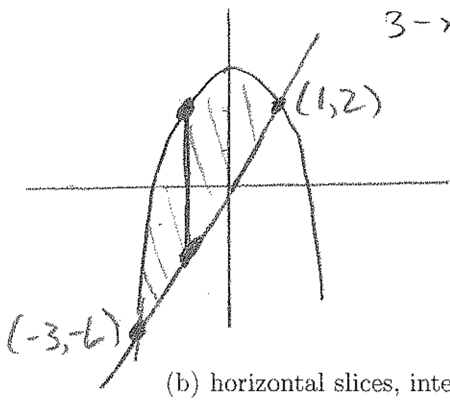
02/13/09 - Buena suerte!

60 points

Calculators may be used, but you must show all work - unsupported answers (e.g., calculator output) will receive minimal credit

1.(8) Let R be the region bounded by $y = 3 - x^2$ and $y = 2x$. (Note: R extends into the first, second, and third quadrants.) Set up, but *do not evaluate*, integral expressions for the area of R, using

(a) vertical slices, integrating with respect to x.



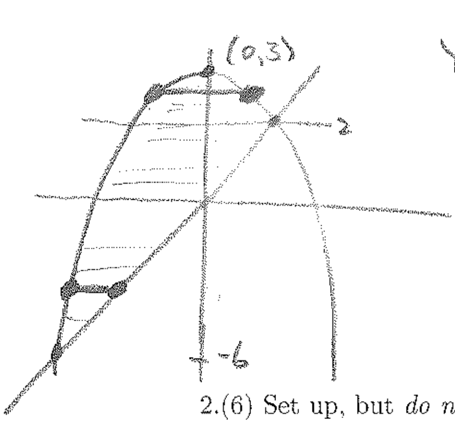
$3 - x^2 = 2x \implies 0 = x^2 + 2x - 3 = (x+3)(x-1)$

$\implies x = -3, 1$

$y = -6, 2$

$$\boxed{A = \int_{-3}^{1} \left[(3 - x^2) - (2x)\right] dx}$$

(b) horizontal slices, integrating with respect to y. (Hint: Draw a large picture.)

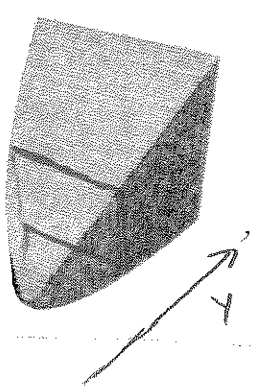


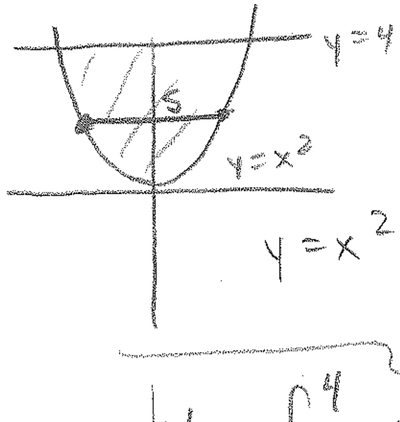
$y = 3 - x^2 \iff x = \pm\sqrt{3-y}$

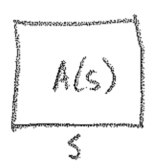
$y = 2x \iff x = \frac{1}{2}y$

$$\boxed{A = \int_{-6}^{2} \left[\tfrac{1}{2}y - (-\sqrt{3-y})\right] dy + \int_{2}^{3} \left[\sqrt{3-y} - (-\sqrt{3-y})\right] dy}$$

2.(6) Set up, but *do not evaluate*, a definite integral for the volume of the solid pictured below. The base of the solid is the region bounded by $y = x^2$ and $y = 4$, and vertical cross-sections perpendicular to the y-axis are squares with one side in the base.







$A(s) = s^2$

$y = x^2 \implies x = \pm\sqrt{y}$

$\implies s = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$

$$\boxed{V = \int_{0}^{4} (2\sqrt{y})^2 \, dy}$$

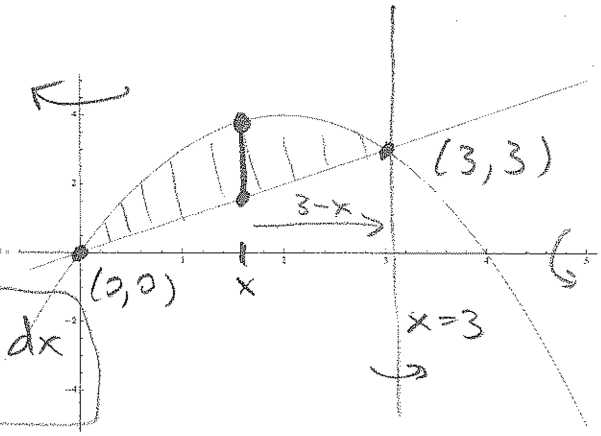
3.(10) Set up, but *do not evaluate* an integral for the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = 4x - x^2$ about

(a) the x-axis.

washers

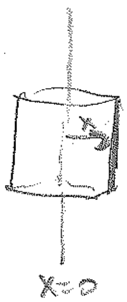
outer radius $= 4x - x^2$

inner radius $= x$



$$\boxed{V = \int_0^3 \pi \left[(4x-x^2)^2 - (x)^2 \right] dx}$$

(b) the y-axis.



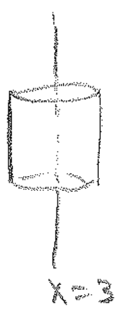
cylindrical shells

radius $= x$

height $= (4x - x^2) - x$

$$\boxed{V = \int_0^3 2\pi x (4x - x^2 - x)\, dx}$$

(c) the vertical line $x = 3$.



cylindrical shells

radius $= 3 - x$

height $= (4x - x^2) - x$

$$\boxed{V = \int_0^3 2\pi (3-x)(4x - x^2 - x)\, dx}$$

4.(8) Find the partial fractions decomposition of the rational function $f(x) = \dfrac{3x}{x^2 + x - 2}$.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3x = A(x-1) + B(x+2)$$

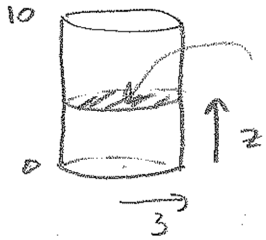
$x=1$ $\quad 3 = A \cdot 0 + B \cdot 3 \implies B = 1$

$x=-2$ $\quad -6 = A \cdot (-3) + B \cdot 0 \implies A = 2$

$$\boxed{\frac{3x}{x^2+x-2} = \frac{2}{x+2} + \frac{1}{x-1}}$$

5.(8) Consider a cylinder of height 10 and radius 3. Set up, but *do not evaluate*, an integral for the mass of the cylinder, if

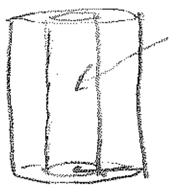
(a) the density at a point is equal to z^2+1, where z is the distance from the point to the (circular) base of the cylinder.



mass = density × area × $dz = (z^2+1)(\pi(3)^2)\,dz$

$$\boxed{\text{Mass} = \int_0^{10} (z^2+1)(\pi\cdot 3^2)\,dz}$$

(b) the density at a point is equal to r^2+1, where r is the distance from the point to the axis of the cylinder.



mass ≈ density × area × $dr = (r^2+1)(2\pi r)(10)\,dr$

$$\boxed{\text{Mass} = \int_0^{3} (r^2+1)(2\pi r)(10)\,dr}$$

6.(10) Evaluate these indefinite integrals. Justify all steps in order to receive credit.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{3}{2}}}\,dx$ (Set $x = 2\tan(u)$.) $\displaystyle = \int \frac{2\sec^2(u)}{8\sec^3(u)}\,du$

$x = 2\tan(u)$

$dx = 2\sec^2(u)\,du$

$x^2+4 = 4\tan^2(u)+4$

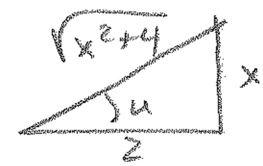
$= 4\sec^2(u)$

$(x^2+4)^{3/2} = 8\sec^3(u)$

$\displaystyle = \frac{1}{4}\int \cos(u)\,du$

$\displaystyle = \frac{1}{4}\sin(u) + C$

$$= \boxed{\frac{x}{4\sqrt{x^2+4}} + C}$$



$\tan(u) = \frac{x}{2}$

$\Rightarrow \sin(u) = \frac{x}{\sqrt{x^2+4}}$

(b) $\displaystyle\int \ln(x^2+1)\,dx$ (Hint: To get started, integrate by parts, with $dv = dx$.)

$u = \ln(x^2+1)$

$dv = dx$

$du = \frac{2x}{x^2+1}\,dx$

$v = x$

$\displaystyle = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx$

$$x^2+1 \overline{)\,2x^2\,} \quad \text{quotient } 2$$
$$\underline{2x^2+2}$$
$$-2$$

$\displaystyle = x\ln(x^2+1) - \int 2 - \frac{2}{x^2+1}\,dx$

$$= \boxed{x\ln(x^2+1) - 2x + 2\tan^{-1}(x) + C}$$

7.(6) (a) Find $\int xe^x\,dx$. $= xe^x - \int e^x\,dx$

$u = x$

$dv = e^x$

$du = dx$

$v = e^x$

$= \boxed{xe^x - e^x + C}$

(b) Derive the reduction formula

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx,$$

for n a positive integer and a a fixed real number.

$u = x^n$

$dv = e^{ax}\,dx$

$du = nx^{n-1}$

$v = \frac{1}{a}e^{ax}$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \int (nx^{n-1})\left(\frac{1}{a}e^{ax}\right)dx$$

$$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \quad \checkmark$$

by integration by parts.

8.(4) Show that $\int_a^b \ln(x)\,dx$ is equal to $\int_{\ln(a)}^{\ln(b)} ue^u\,du$, by changing variables ("pulling back") via the substitution $u = \ln(x)$. Here $0 < a < b < \infty$.

$u = \ln(x) \implies x = e^u$

$dx = e^u\,du$

$x = a \implies u = \ln(a)$

$x = b \implies u = \ln(b)$

so $\int_a^b \ln(x)\,dx = \int_{\ln(a)}^{\ln(b)} (u)(e^u\,du)$

$$= \int_{\ln(a)}^{\ln(b)} ue^u\,du \quad \checkmark$$