Homotopy Properties of the Poset of Nontrivial $p$-Subgroups of a Group

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Let $G$ be a group and let $\mathcal{P}_p(G)$ be the set of nontrivial $p$-subgroups of $G$ ordered by inclusion, where $p$ is a prime. To the poset (= partially ordered set) $\mathcal{P}_p(G)$ one can associate a simplicial complex $|\mathcal{P}_p(G)|$ in a well-known way. This simplicial complex appears in the work of Brown [2, 3] on Euler characteristics and cohomology for discrete groups. One of his results is an interesting variant of the Sylow theorem on the number of Sylow groups; it asserts that for $G$ finite the Euler characteristic of $|\mathcal{P}_p(G)|$ is congruent to 1 modulo the order of a Sylow $p$-subgroup.

Our aim in this paper is to investigate various homotopy invariants of the simplicial complex $|\mathcal{P}_p(G)|$, such as its homology, connectivity, etc. We now review the contents of the paper.

In Section 1 we review properties of the functor $X \mapsto |X|$. Throughout the paper we use this functor to assign topological concepts to posets. For example, we call two posets homotopy equivalent when the associated simplicial complexes are.

In Section 2 we first show that $\mathcal{P}_p(G)$ is homotopy equivalent to the poset $\mathcal{A}_p(G)$ consisting of elementary abelian $p$-groups (called $p$-tori for short). Hence the remainder of the paper is mostly concerned with the smaller poset $\mathcal{A}_p(G)$. We show $\mathcal{A}_p(G_1 \times G_2)$ is homotopy equivalent to the join of $\mathcal{A}_p(G_1)$ and $\mathcal{A}_p(G_2)$. We prove that $\mathcal{A}_p(G)$ is contractible when $G$ has a nontrivial normal $p$-subgroup, and state a conjecture to the effect that the converse holds when $G$ is finite. The conjecture is proved for solvable groups in Section 12; the proof served as motivation for most of the second half of the paper.

In Section 3 we show for a finite Chevalley group that $\mathcal{A}_p(G)$ has the homotopy type of the Tits building associated to $G$, and hence it has the homotopy type of a bouquet of spheres. Section 4 contains a proof of the aforementioned result of Brown. The next two sections relate the connected components of $\mathcal{P}_p(G)$ to topics in finite group theory, e.g., in Section 6, Puig's analysis [8] of the Alperin fusion theorem is described.

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The second half of the paper is concerned with showing that for certain groups G, the poset $\mathcal{A}_p(G)$ has the homotopy type of a bouquet of $d$-spheres, where $d$ is its dimension, in which case we say it is $d$-spherical. In fact, for the groups considered the poset $\mathcal{A}_p(G)$ has the stronger property of being Cohen–Macaulay in the sense of combinatorial theory [10, 12], that is, not only is it spherical, but the link at each point of the simplicial complex is $(d-1)$-spherical.

In Section 7 we collect standard facts about the homology and fundamental group of a poset, in particular, the homology spectral sequence of a map. Sections 8 and 9 discuss spherical and Cohen–Macaulay (CM for short) posets. We prove a useful result showing a poset $X$ is CM when there is a map $f$ to a CM poset $Y$ whose “fibers” $f/y$ are CM of the appropriate dimension.

In Sections 10, 11 we consider groups such that $\mathcal{A}_p(G)$ is CM. We show the class of these groups is closed under products and also by extensions by a solvable group which is uniquely $p$-divisible in a suitable sense. These results are applied in Section 12 to prove the conjecture of Section 2 for solvable groups, and to prove $\mathcal{A}_p(\text{GL}_n(F))$ is CM when $p$ is different from the characteristic of the field $F$ and $F$ contains a primitive $p$th root of unity.

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1. The Simplicial Complex Associated to a Poset

Let $X$ be a partially ordered set (poset, for short). Let $|X|$ denote the simplicial complex associated to $X$, that is, the simplicial complex whose vertices are the elements of $X$ and whose simplices are the nonempty finite chains in $X$. If $X$ is viewed as a small category, then $|X|$ is the classifying space of $X$ (see [9, 11]). Many of the properties of the construction $X \mapsto |X|$ reviewed below hold more generally for the classifying space construction.

If $X^{\text{op}}$ denotes the poset dual to $X$, i.e., with the opposite ordering, then one has an isomorphism of simplicial complexes

$$|X^{\text{op}}| \cong |X|.$$  

(1.1)

By a morphism (or map) of posets $f: X \to Y$, we mean an order-preserving map $(x' \leq x \Rightarrow f(x') \leq f(x))$. Such a map induces a simplicial map $|f|: |X| \to |Y|$. If $X \times Y$ is the product of the posets $X$, $Y$, one has a homeomorphism

$$|X \times Y| \cong |X| \times |Y|$$  

(1.2)

induced by maps $|pr_1|$. We note that the simplicial complex associated to a poset is equipped with the weak topology, hence the product on the right has to be taken in the category of compactly generated spaces.
1.3. **Homotopy property.** If \( f, g : X \to Y \) are maps of posets such that \( f(x) \leq g(x) \) for all \( x \) in \( X \), then \(|f|\) and \(|g|\) are homotopic.

(This follows from the product formula (1.2) by noting that \( f, g \) determine a map \( \{0 < 1\} \times X \to Y \) and that \(|\{0 < 1\}|\) is a 1-simplex.)

If \( K \) is a simplicial complex, its barycentric subdivision is by definition the simplicial complex \( |S(K)| \), where \( S(K) \) is the poset of simplices in \( K \) ordered by inclusion. Hence one has a homeomorphism

\[
|S(K)| = K. \tag{1.4}
\]

The construction \( X \mapsto |X| \) allows us to assign topological concepts to posets. For example, we define the homology groups of \( X \) to be those of \(|X|\), and we call \( X \) contractible if \(|X|\) is contractible.

1.5. A poset \( X \) will be called **conically contractible** if there is an \( x_0 \) in \( X \) and a map of posets \( f : X \to X \) such that \( x \leq f(x) > x_0 \) for all \( x \) in \( X \) (e.g., if the join \( x \lor x_0 \) exists for all \( x \), one can take \( f(x) = x \lor x_0 \)). In this case the homotopy property shows that the maps \(|id_X|\), \(|f|\), and the constant map with value \( x_0 \) from \(|X|\) to itself are homotopic, and hence \( X \) is contractible. This terminology comes from the example \( X = S(K) \), where \( K = \text{C}(L) \) is the cone on the simplicial complex \( L \).

The homotopy property can be used to show a map \( f : X \to Y \) is a homotopy equivalence (i.e., \(|f|\) is a homotopy equivalence) when there exists a map \( g : Y \to X \) with suitable properties (e.g., \( f, g \) are adjoint considered as functors between categories). However, we need a more powerful way of showing a map is a homotopy equivalence.

Given a map \( f : X \to Y \) of posets and an element \( y \) of \( Y \), we define subposets of \( X \) as follows:

\[
\begin{align*}
\mathcal{f}/y &= \{ x \in X \mid f(x) \leq y \}, \\
\mathcal{y}/f &= \{ x \in X \mid f(x) \geq y \}.
\end{align*}
\]

**Proposition 1.6.** Assume \( \mathcal{f}/y \) is contractible for all \( y \) in \( Y \) (resp. \( \mathcal{y}/f \) is contractible for all \( y \)). Then \( f \) is a homotopy equivalence.

For a proof see [9, Theorem A] and also the end of Section 7.

A subset \( S \) of a poset \( X \) will be called **closed** if it is closed under specialization, that is, if \( x' \leq x \in S \) implies \( x' \in S \). Let \( \text{Cl}(X) \) be the poset of closed subsets of \( X \) ordered by inclusion.

Let \( X, Y \) be posets, let \( Z \) be a closed subset of \( X \times Y \), and let \( p_1 : Z \to X \), \( p_2 : Z \to Y \) denote the maps induced by the projections. The fiber of \( p_1 \) over a point \( x \) of \( X \) can be identified with the subposet \( Z_x = \{ y \in Y \mid (x, y) \in Z \} \) of \( Y \). Clearly \( Z_x \) is a closed subset of \( Y \) and \( x \mapsto Z_x \) is a contravariant (i.e., order-reversing) map from \( X \) to \( \text{Cl}(Y) \). One shows easily that in this way one obtains a one–one correspondence between closed subsets of \( X \times Y \) and
contravariant maps from \( X \) to \( \text{Cl}(Y) \). Similarly \( y \mapsto Z_y = \{ x \in X \mid (x, y) \in Z \} \) is a contravariant map from \( Y \) to \( \text{Cl}(X) \) which determines and is determined by \( Z \).

**Proposition 1.7.** If \( Z_x \) is contractible for each \( x \) in \( X \), then \( p_1: Z \to X \) is a homotopy equivalence.

The poset \( x \backslash p_1 \) consists of all \((x', y)\) in \( Z \) with \( x' \geq x \). Let \( u: Z_x \to x \backslash p_1 \) and \( v: x \backslash p_1 \to Z_x \) be the maps given by \( u(y) = (x, y) \), \( v(x', y) = y \). One has \( v u(y) = y \) and \( u v(x', y) \leq (x', y) \), hence \( u \) and \( v \) are homotopy inverses by 1.3. Since \( Z_x \) is contractible, so is \( x \backslash p_1 \), hence \( p_1 \) is a homotopy equivalence by 1.6.

**Corollary 1.8.** If \( Z_x \) and \( Z_y \) are contractible for each \( x \) in \( X \) and \( y \) in \( Y \), then \( X \) and \( Y \) are homotopy equivalent.

In effect, both \( p_1: Z \to X \) and \( p_2: Z \to Y \) are homotopy equivalences by the proposition.

We conclude this section by describing two ways of realizing the join \( |X * Y| \) of the spaces \( |X| \) and \( |Y| \) by posets.

We define the cone \( CX \) of a poset \( X \) to be the poset obtained by adjoining to \( X \) an element \( 0 \) smaller than all the elements of \( X \). Clearly \( |CX| \) is the cone on the simplicial complex \( |X| \). We define the join \( X * Y \) of the posets \( X \) and \( Y \) to be the disjoint union of \( X \) and \( Y \) equipped with the ordering which agrees with the given orderings on \( X \) and \( Y \) and which is such that any element of \( X \) is less than any element of \( Y \). For example, \( CX = \{0\} * X \).

Consider the product \( |CX| \times |CY| = |CX \times CY| \). The join \( |X * Y| \) is the union of the line segments in the product joining a point of \( |X| \) to a point of \( |Y| \). It is the simplicial complex whose simplices are nonempty subsets of the form \( \sigma \coprod \tau \), where \( \sigma \) (resp. \( \tau \)) is a possibly empty subset in \( X \) (resp. \( Y \)). Consequently, one has a homeomorphism

\[
|X * Y| = |X| * |Y|
\]

relating the join operations on posets and spaces.

On the other hand, radial projection from the vertex \((0, 0)\) in \( |CX \times CY| \)
furnishes a homeomorphism

\[ |X \ast Y| \cong |CX \times Y| \cup |X \times CY| = |CX \times Y \cup X \times CY| = |CX \times CY - \{(0, 0)\}|.\]

Summarizing, we have proved

**Proposition 1.9.** There are canonical homeomorphisms

\[ |X \ast Y| = |X| \ast |Y| = |CX \times CY - \{(0, 0)\}|.\]

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### 2. The Posets \( \mathcal{S}_p(G) \) and \( \mathcal{A}_p(G) \)

Let \( p \) be a prime number. By a \( p \)-group we mean a finite group of order a power of \( p \). By a \( p \)-torus we mean a finite abelian group \( A \) whose elements have order 1 or \( p \); the rank of \( A \), denoted \( r_p(A) \), is the dimension of \( A \) considered as a vector space over \( \mathbb{F}_p \).

If \( G \) is a group, we denote by \( \mathcal{S}_p(G) \) the poset of nonidentity \( p \)-subgroups of \( G \) ordered by inclusion, and we let \( \mathcal{A}_p(G) \) be the subposet consisting of all nonidentity \( p \)-tori in \( G \). The following proposition shows that, as far as homotopy invariants are concerned, one can replace \( \mathcal{S}_p(G) \) by the smaller poset \( \mathcal{A}_p(G) \).

**Proposition 2.1.** The inclusion \( \mathcal{A}_p(G) \subset \mathcal{S}_p(G) \) is a homotopy equivalence.

**Proof.** If this inclusion is denoted by \( i \), then one has \( i/P = j(P) \) for any \( P \) in \( \mathcal{S}_p(G) \). By 1.6, the map \( i \) is a homotopy equivalence provided each \( i/P \) is contractible, so it suffices to prove the following.

**Lemma 2.2.** If \( P \) is a nontrivial \( p \)-group, then \( \mathcal{A}_p(P) \) is contractible.

If \( B \) is the subgroup of the center of \( P \) consisting of the elements of order 1 or \( p \), one knows that \( B > 1 \), since \( P > 1 \). Hence in \( \mathcal{A}_p(P) \) one has \( A \leq AB \geq B \) for all \( A \), so \( \mathcal{S}_p(P) \) is conically contractible in the sense of 1.5. Q.E.D.

By the *dimension* of a poset \( X \) we mean the dimension of the simplicial complex \( |X| \), or equivalently, the supremum of the integers \( n \) such that there is a chain \( x_0 < \cdots < x_n \) in \( X \). By convention the empty set has dimension \(-1\).

By the *\( p \)-rank* \( r_p(G) \) of the group \( G \), we mean the supremum of the ranks of the \( p \)-tori in \( G \). Since any \( p \)-torus of rank \( r \) has a composition series of length \( r \), one sees easily that one has

\[ \dim \mathcal{A}_p(G) = r_p(G) - 1. \tag{2.3} \]

This has the consequence that the homology groups of \( \mathcal{A}_p(G) \), which are the same as those of \( \mathcal{S}_p(G) \) by 2.1, vanish in degrees \( \geq r_p(G) \).
Proposition 2.4. If $G$ has a nontrivial normal $p$-subgroup, then $A_p(G)$ is contractible.

It suffices to show $I_p(G)$ is contractible by 2.1. However, if $Q$ is a nontrivial normal $p$-subgroup of $G$, then in $I_p(G)$ one has $P \leq PQ \geq Q$ for all $P$, so $I_p(G)$ is conically contractible.

Remark 2.5. As in the proof of 2.1, one can use 2.4 to show that $I_p(G)$ is homotopy equivalent to the poset consisting of those subgroups $H$ of $G$ which contain a nontrivial normal (in $H$) $p$-subgroup.

Proposition 2.6. The poset $I_p(G \times G)$ is homotopy equivalent to the join of $I_p(G_1)$ and $I_p(G_2)$.

Proof. Let $T$ be the subset of $I_p(G_1 \times G_2)$ consisting of subgroups of the form $A_1 \times A_2$, where $A_1$ is a $p$-torus in $G_1$ and not both $A_1$ and $A_2$ are the identity. Then $T$ can be identified with the poset $C \times I_p(G_2) - \{(0,0)\}$ of 1.9, which by this proposition has the homotopy type of the join $I_p(G_1) * I_p(G_2)$. Hence it suffices to show that the inclusion map $i: T \to I_p(G_1 \times G_2)$ is a homotopy equivalence. But one has a map $r$ going the other way given by $r(A) = pr_1(A) \times pr_2(A)$, where the $pr_j$ are the projections of $G_1 \times G_2$ on its factors. One has $ri = id$ and $A \leq i(r(A))$ for all $A$ in $I_p(G_1 \times G_2)$, hence by the homotopy property 1.3 the maps $r$ and $i$ are homotopy inverses, proving the proposition.

Example 2.7. Let $G = \mathfrak{S}_n$ be the symmetric group of degree $n$ and let $p$ be an odd prime. For $n < p$, $G$ is of order prime to $p$, hence $I_p(G)$ is empty in this case. For $p \leq n < 2p$, the Sylow $p$-subgroups of $G$ are cyclic of order $p$, and there are $n!/p(p-1)(n-p)!$ of them, hence $I_p(G)$ is a zero-dimensional poset with this many elements.

Suppose $n = 2p$. If an element of $G$ has order $p$, it has at least one orbit in the set $\{1, \ldots, 2p\}$ with $p$-elements. It follows that any $A$ in $I_p(G)$ leaves invariant a unique partition into two disjoint subsets of cardinality $p$. Thus $I_p(G)$ decomposes into a disjoint union indexed by these partitions. Since these partitions are transitively permuted by $G$, the parts of the decomposition are isomorphic, the isomorphism between any two of them being given by conjugation with respect to an element of $G$. The part of $I_p(G)$ belonging to the partition $\{1, \ldots, p\} \sqcup \{p+1, \ldots, 2p\}$ is the subposet $I_p(H)$, where $H = (\mathfrak{S}_p)^2 \rtimes \mathfrak{S}_2$ is the stabilizer of this partition. Using the fact $p$ is odd and 2.6, we have

$$I_p(H) = I_p(\mathfrak{S}_p^2) \sim I_p(\mathfrak{S}_p) * I_p(\mathfrak{S}_p)$$

where $I_p(\mathfrak{S}_p)$ is the disjoint union of $(p-2)!$ points. Therefore we see that $I_p(\mathfrak{S}_p)$ is a disconnected graph with $2p)!/2(p!)^2$ components, each of which
has the homotopy type of a bouquet of \(((p - 2)! - 1)^2\) circles. Hence for 
\(p \geq 5\), \(\mathcal{A}_p(\mathbb{S}_{2p})\) has nontrivial homology in degrees 0 and 1.

We note that the preceding calculation also applies to the alternating groups 
\(A_n, n \leq 2p\), since \(\mathcal{A}_p(A_n) = \mathcal{A}_p(\mathbb{S}_p)\) for \(p\) odd.

To conclude this section we discuss a conjecture concerning the converse of 2.4 for finite groups.

First we note that, quite generally, if a group \(G\) acts on a poset \(X\), then one has the formula

\[ |X|^\circ = |X^\circ| \]  (2.8)

relating the fixpoints for the action on \(X\) and on the simplicial complex \(|X|\). In effect, because of the ordering, a simplex of \(|X|\) is carried into itself by an element \(g\) if and only if all the vertices of the simplex are fixed under \(g\).

Consider now the \(G\)-action on \(\mathcal{A}_p(G)\) induced by inner automorphisms of \(G\). Here \(\mathcal{A}_p(G)^\circ\) is the poset of nonidentity normal \(p\)-tori in \(G\), so (2.8) implies \(|\mathcal{A}_p(G)|^G \neq \emptyset\) if and only if \(G\) has a nontrivial normal \(p\)-torus. On the other hand, if \(H\) is a nontrivial normal \(p\)-subgroup of \(G\), then the subgroup \(B\) of elements of order 1 or \(p\) in the center of \(H\) is characteristic in \(H\), hence \(B\) is a normal \(p\)-torus in \(G\). Thus 2.4 can be reformulated as saying that if \(|\mathcal{A}_p(G)|^G\) has a \(G\)-fixpoint, then it is contractible.

It is an interesting problem whether the converse of 2.4 holds. For \(G\) finite there is evidence for the following.

**Conjecture 2.9.** If \(G\) is finite and \(\mathcal{A}_p(G)\) is contractible, then \(G\) has a nontrivial normal \(p\)-subgroup.

We will show that this conjecture is true for solvable groups in part II. One also has the following special case.

**Proposition 2.10.** The above conjecture holds if \(r_p(G) \leq 2\).

**Proof.** We can rule out the case \(r_p(G) = 0\), for then \(\mathcal{A}_p(G)\) is empty, hence not contractible. If \(r_p(G) = 1\), then \(\mathcal{A}_p(G)\) is zero dimensional and contractible, hence it is a single point necessarily fixed under \(G\), so \(G\) has a nontrivial normal \(p\)-torus. Finally, if \(r_p(G) = 2\), then \(|\mathcal{A}_p(G)|\) is a one-dimensional contractible simplicial complex, that is, it is a tree. But one knows (Serre) that a finite group acting on a tree always has a fixpoint, so \(G\) has a nontrivial normal \(p\)-torus in this case also. 

Q.E.D.

### 3. Relation to Buildings

Let \(G\) be the finite groups of rational points of a semisimple algebraic group defined over a finite field \(k\), and let \(p\) be the characteristic of \(k\). Let \(T\) be the building associated to \(G\) in the sense of Tits [13]. It is a simplicial complex
of dimension $l - 1$ on which $G$ operates, where $l$ is the rank of the underlying algebraic group over $k$. By associating to a simplex of $T$ its stabilizer in $G$ one obtains a contravariant isomorphism between the poset of simplices in $T$ and the poset of proper "parabolic" subgroups of $G$.

For example, if $G = SL_n(k)$, then the building $T$ can be identified with the simplicial complex $|T(k^n)|$, where $T(k^n)$ denotes the poset of proper subspaces of the vector space $k^n$. Here $l = n - 1$, which is quite different from $r_p(G)$.

**Theorem 3.1.** The poset $\mathcal{A}'(G)$ is homotopy equivalent to the building $T$. Consequently $\mathcal{A}'(G)$ has the homotopy type of a bouquet of $(l - 1)$-dimensional spheres.

*Proof.* From the theory of buildings [13] one knows that $T$ has the homotopy type of a bouquet of $(l - 1)$-spheres, hence we only have to show that $\mathcal{A}'(G)$ and $T$ are homotopy equivalent. Let $X$ be the poset of simplices in $T$ and let $G_x$ denote the subgroup of $G$ carrying the simplex $x$ into itself. Then $|X|$ is the barycentric subdivision of $T$, so it will suffice to show the posets $X$ and $\mathcal{A}'(G)$ are homotopy equivalent. Let $Z$ be the subset of $X \times \mathcal{A}'(G)$ consisting of $(x, A)$ such that $x \in X^A$, or equivalently $A \subseteq G_x$. One knows that an element of $G_x$ leaves each point of the simplex $x$ fixed, hence $x' \leq x \Rightarrow G_{x'} \subseteq G_x$, and so $Z$ is closed. By 1.8 we can conclude $X$ and $\mathcal{A}'(G)$ are homotopy equivalent once we show the posets $Z_x = \mathcal{A}'(G_x)$ and $Z_A = X^A$ are contractible. Because of 2.4 and (2.8) it suffices to verify the following assertions:

(a) The group $G_x$ has a nontrivial normal $p$-subgroup;

(b) if $H$ is a nontrivial $p$-subgroup of $G$, then $T^H$ is contractible.

The group $G_x$ is the group of rational points of a proper parabolic subgroup of the algebraic group underlying $G$. From the theory of roots, one knows the unipotent radical of this parabolic subgroup is nontrivial. The group of rational points of this unipotent radical is a nonidentity normal $p$-subgroup of $G_x$, proving (a).

To prove (b) we use a variant of the Tits argument showing $T$ has the homotopy type of a bouquet of spheres. We choose a chamber of $T$ with $H \subseteq G_x$. This is possible, for the groups $G_x$ as $x$ runs over the chambers are the normalizers of the different Sylow $p$-subgroups of $G$. If $x'$ is a chamber opposite to $x$, then $G_{x'} \cap G_x$ has order prime to $p$, so $x'$ is not fixed under $H$. It follows that if $c$ is an interior point of $x$, then $T^H$ contains no points of $T$ opposite to $c$, hence there is a unique "geodesic" in $T$ joining a given point of $T^H$ to $c$. By uniqueness this geodesic lies in $T^H$, hence $T^H$ contracts to $c$ along geodesics, proving assertion (b). Q.E.D.

In the case of $SL_n(k)$ the assertions (a) and (b) can be proved in elementary fashion as follows. A simplex $x$ of $T = |T(k^n)|$ is a flag $0 < V_1 < \cdots < V_r < V$...
in the vector space $V = k^n$, where $r \geq 1$, and $G_x$ is the stabilizer of this flag. The elements of $G_x$ which induce the identity on each quotient of the flag form a nontrivial normal $p$-subgroup of $G_x$, whence (a). As for (b) we note that $T'$ is the simplicial complex associated to the poset $T(V)'$ of proper $H$-invariant subspaces of $V$. Because $H$ is a $p$-group, one has $W'H > 0$ for $W \in T(V)'$, hence this poset is conically contractible: $W \geq W'H \leq V'$, proving (b).

4. Euler Characteristic of $\mathcal{L}_p(G)$

In this section we present results of Brown [2] specialized to the case of finite groups.

Let $G$ be a finite group whose order is divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. We consider the restriction of the action of $G$ on $| \mathcal{L}_p(G) |$ given by conjugation to the subgroup $P$.

**Proposition 4.1.** The subspace of $| \mathcal{L}_p(G) |$ consisting of points where the action of $P$ is not free is contractible.

**Proof.** We replace $| \mathcal{L}_p(G) |$ by its barycentric subdivision $| Y |$, where $Y$ is the poset of simplices in $| \mathcal{L}_p(G) |$. Then the subspace where the $P$-action is not free is the subcomplex

$$\bigcup_H | Y'_H | = \bigcup_H | Y'H | = | Y' |$$

where $Y' = \bigcup Y'H$ and $H$ ranges over the poset $\mathcal{L}_p(P)$ of nontrivial subgroups of $P$. Let $Z \subset \mathcal{L}_p(P) \times Y'$ be the closed subset consisting of pairs $(H, y)$, where $y \in Y'H$, or equivalently, $H$ is contained in the isotropy group $P_y$. We are going to apply 1.8 to $Z$ to conclude $Y'$ and $\mathcal{L}_p(P)$ are homotopy equivalent; as $\mathcal{L}_p(P)$ has the largest element $P$, it is contractible, hence $Y'$ will be contractible, proving the proposition.

By the definition of $Y'$, the group $P_y$ is a nontrivial $p$-group for any $y$ in $Y'$, hence $Z_y = \mathcal{L}_p(P_y)$ is contractible. On the other hand, if $H \in \mathcal{L}_p(P)$, then $Z_H = Y'H$, which is the poset of simplices in $| \mathcal{L}_p(G)'H |$, where $\mathcal{L}_p(G)'H$ consists of nontrivial $p$-subgroups of $G$ normalized by $H$. The poset $\mathcal{L}_p(G)'H$ is conically contractible: $Q \leq QH \geq H$, so $| Z_H | = | Y'H | = | \mathcal{L}_p(G)'H |$ is contractible, concluding the proof.

**Corollary 4.2.** The Euler characteristic $\chi(| \mathcal{L}_p(G) |)$ is $\equiv 1$ modulo the order of the Sylow group $P$.

**Proof.** In the notation of the preceding proof one has

$$\chi(| Y |) = 1 + \chi(| Y' | Y' |)$$
because \(|Y'|\) is contractible. But the group \(P\) acts freely on the simplices of \(|Y|\) outside of \(|Y'|\), hence \(\chi(|Y|, |Y'|) = 0\) modulo the order of \(P\), proving the corollary.

This corollary can be generalized as follows. Let \(k = \mathbb{F}_p\) and let \(R_k(G)\) be the Grothendieck group of finitely generated \(k[G]\)-modules. If \([V] \in R_k(G)\) denotes the class of the module \(V\), then the differences \([P] - [Q]\), with \([P]\) and \([Q]\) finitely generated projective modules, form an ideal in \(R_k(G)\). If \(\tilde{C}((K, k)\) and \(\tilde{H}_i(K, k)\) denote the reduced chain complex and homology of the finite simplicial complex \(K\) with coefficients in \(k\), one knows that

\[
\sum_i (-1)^i [\tilde{C}_i(K, k)] = \sum_i (-1)^i [\tilde{H}_i(K, k)]
\]

in \(R_k(G)\).

**Corollary 4.3.** If \(K = |\mathcal{S}_p(G)|\), then the above element of \(R_k(G)\) lies in the ideal of differences of classes of projective modules.

**Proof.** Instead of \(K\) we work with \(|Y|\). One easily constructs an exact sequence of complexes of finitely generated \(k[G]\)-modules

\[
0 \rightarrow M_i \rightarrow F_i \rightarrow \tilde{C}_i((|Y|, k) \rightarrow 0
\]

where \(F_i\) is a complex of free modules zero outside degrees \(0 \leq i \leq n\), and where \(M_i\) has zero homology in degrees \(\neq n\). As the element of \(R_k(G)\) associated to a complex adds with respect to exact sequences, we have only to show \(H_n(M_i)\) is a projective \(k[G]\)-module. One knows it suffices to show \(H_n(M_i)\) is projective over \(k[P]\). Let \(M.'\) be the kernel of the composition

\[
F_i \rightarrow \tilde{C}_i((|Y|, k) \rightarrow C_i((|Y|, |Y'|, k).
\]

Then \(M.'\) is a complex of projective \(k[P]\)-modules, because \(P\) acts freely on the simplices of \(|Y|\) outside of \(|Y'|\). One has an exact sequence

\[
0 \rightarrow M_i \rightarrow M_i' \rightarrow \tilde{C}_i((|Y'|, k) \rightarrow 0
\]

where \(|Y'|\) is contractible, so \(M_i\) and \(M_i'\) have the same homology. Hence we have an exact sequence

\[
0 \rightarrow H_n(M_i) \rightarrow H_n(M_i') \rightarrow \cdots \rightarrow M_0' \rightarrow 0
\]

from which it follows that \(H_n(M_i)\) is projective over \(k[P]\), concluding the proof.

**Remarks.**

4.4. All the results of this section remain valid with \(\mathcal{S}_p(G)\) replaced by \(\mathcal{A}_p(G)\). To see that \(\mathcal{A}_p(G)^H\) is contractible for \(H\) in \(\mathcal{S}_p(G)\), we note that
if a nontrivial $p$-torus $A$ is normalized by $H$, then the centralizer $A^H$ is non-trivial. Hence if $B$ is the $p$-torus of central elements of order dividing $p$ in $H$ one has $A \supseteq A^H \subseteq A^H B \supseteq B$, and it follows easily that $\mathcal{S}_p(G)^H$ is contractible.

4.5. The proof of 4.3 shows that the complex $\tilde{C}_i(\mathcal{S}_p(G), k)$ is a "perfect" complex of $kG$-modules. One can prove this implies that if all the groups $H_i(\mathcal{S}_p(G), k)$ are zero except the $i$th one, then the $i$th group is a projective $k[G]$-module. In the situation considered in Section 3, where $\mathcal{S}_p(G)$ was identified with the Tits building up to homotopy, Lusztig [7] has shown that the class in $R_k(G)$ of this unique nonzero homology group generates the ideal of projective classes. However, examples of Feit [4] show this theorem of Lusztig is somewhat special to Chevalley groups.

5. CONNECTED COMPONENTS OF $\mathcal{S}_p(G)$

In this section the group $G$ is assumed to be finite of order divisible by $p$, and $\text{Syl}_p(G)$ denotes the set of Sylow $p$-subgroups of $G$.

By the connected components of a poset we mean the equivalence classes for the equivalence relation generated by the order relation. Thus two elements are in the same component when they are joined by a sequence whose successive members are comparable. We denote by $\pi_0X$ the set of connected components of the poset $X$. These are in one-one correspondence with connected components of the simplicial complex $|X|$, hence $\pi_0X \cong \pi_0|X|$.

We consider the set $\pi_0\mathcal{S}_p(G)$, which can be identified with $\pi_0\mathcal{G}_p(G)$ by 2.1. Every $p$-subgroup is contained in a Sylow subgroup, hence every connected component of $\mathcal{S}_p(G)$ contains at least one Sylow subgroup. Hence $G$ acts transitively on $\pi_0\mathcal{S}_p(G)$, so we have

$$G/N \cong \pi_0\mathcal{S}_p(G)$$

(5.1)

where $N$ is the stabilizer of some component. We wish to determine $N$.

**Proposition 5.2.** The following conditions on a subgroup $M$ of $G$ are equivalent:

(a) There is a component of $\mathcal{S}_p(G)$ such that $M$ contains the stabilizer in $G$ of this component.

(b) There is a $P \in \text{Syl}_p(G)$ such that $M \supset N_C(H)$ for all $H$ in $\mathcal{S}_p(P)$.

(c) The subgroup $M$ contains $N_C(P)$ for some $P$ in $\text{Syl}_p(G)$, and for any $p$-subgroup $K$ of $G$ such that $K \cap M > 1$, one has $K \subseteq M$.

(d) $p$ divides the order of $M$ and $M \cap M^x$ is of order prime to $p$ for all $x \notin M$.

Here $M^x = x^{-1}Mx$ and $N_C(H)$ is the normalizer of the subgroup $H$. 
Proof. (a) ⇒ (b) Let $P$ be a Sylow subgroup in the component $X$. If $g \in N_G(H)$ for some $H$ in $\mathcal{S}_p(P)$, then $X$ and $X^g$ are components containing $H$, so $X = X^g$ and $g \in M$.

(b) ⇒ (c) Clearly $M \supset N_G(P)$. Note that because any $p$-subgroup $H$ of $M$ is conjugate in $M$ to a subgroup of $P$, we have $H > 1$ implies $N_G(H) \subset M$. Suppose given a $p$-subgroup $K$ of $G$ with $K \cap M > 1$; if $K \cap M < K$, one has because $K$ is a $p$-group

$$K \cap M < N_K(K \cap M) = K \cap N_G(K \cap M) \subset K \cap M$$

which is a contradiction. Thus $K \subset M$.

(c) ⇒ (d) Clearly $p$ divides the order of $M$. Assume $M \cap M^p$ contains a nontrivial $p$-group $H$ and let $Q \in \text{Syl}_p(G)$ contain $H$. By (c), one has $Q \subset M$, and also $Q^{p^2} \supset H^{p^2} \subset M$, so $Q^{p^2} \subset M$. Since $P$, $Q$, $Q^{p^2}$ are Sylow groups of $M$, one has $Q = P^m$ and $Q^{p^2} = P^{m'}$ with $m$, $m'$ in $M$; hence $mx^{-1}(m')^{-1} \in N_G(P) \subset M$, so $x \in M$. This proves (d).

(d) ⇒ (a) We have

$$(G : M) = \sum_{M \in M} (M : M \cap M^g) \equiv 1 \pmod{p},$$

hence $M$ contains a Sylow group $P$ of $G$. Let $N$ be the subgroup of $G$ leaving invariant the component of $\mathcal{S}_p(G)$ containing $P$. If $n \in N$, then $P$ and $P^n$ are in the same component, hence it is easily seen that there is a sequence of Sylow groups $P = P_0, P_1, \ldots, P_s = P^n$ such that $P_{i-1} \cap P_i > 1$. Let $x_i$ be such that $(P_{i-1})^{x_i} = P_i$ for $1 \leq i \leq m$; we can choose $x_i$ so that $x_1 \cdots x_m = n$. Assuming $P_{i-1} \subset M$ we have $1 < P_{i-1} \cap P_i \subset M \cap M^{x_i}$, so by (d) we have $x_i \in M$ and $P_i \subset M$ for all $i$. Hence $n \in M$, so $N \subset M$.

Q.E.D.

**Corollary 5.3.** If $N$ is the subgroup of $G$ leaving invariant the component of $\mathcal{S}_p(G)$ containing $P$, then $N$ is generated by $N_G(H)$ for $1 < H \subset P$.

This is clear.

We next consider groups such that $\mathcal{S}_p(G)$ is disconnected, or equivalently by the above, such that there exists a subgroup $M < G$ with the properties in 5.2. An example is furnished by a group having $p$-rank 1 and having no nontrivial normal $p$-subgroup, for in this case $\mathcal{S}_p(G)$ is zero dimensional with more than one element. We shall now show that for $p$-solvable groups these are the only examples.

As usual, let $O_{p'}(G)$ be the largest normal $p'$-subgroup and $O_{p',s}(G)/O_{p'}(G)$ be the largest normal $p$-subgroup of $G/O_{p'}(G)$.

**Proposition 5.4.** Assume $O_{p',r}(G)$ $> O_{p'}(G)$. If $\mathcal{S}_p(G)$ is not connected, then the $p$-rank of $G$ is 1.
Put $H = O_p'(G)$, $K = O_{p'}(G)$. If $Q \in \text{Syl}_p(K)$, then $K = QH$, so $H$ acts transitively on $\pi_0\mathcal{X}_p(K)$, because any Sylow group of $K$ is conjugate to $Q$ by an element of $H$. Since $P \in \text{Syl}_p(G)$ implies $P \cap K \in \text{Syl}_p(K)$, it follows that $\pi_0\mathcal{X}_p(K)$ maps onto $\pi_0\mathcal{X}_p(G)$, hence $H$ acts transitively on the latter. Let $A$ be a maximal $p$-torus of $G$. The map from $\pi_0\mathcal{X}_p(\text{HA})$ to $\pi_0\mathcal{X}_p(G)$ is surjective because $H$ acts transitively, hence $\mathcal{X}_p(\text{HA})$ is not connected. If $X$ is the component of $\mathcal{X}_p(G)$ containing $A$, it is clear that $X$ is left fixed by the centralizers $C_H(B)$ for all $1 < B \subset A$. But, by Gorenstein [6, (6.2.4)], $H$ is generated by these centralizers if $A$ is not cyclic, hence in this case $X$ would be fixed by $H$ which is a contradiction. Thus $A$ is cyclic, proving the proposition.

The example of $SL_2(F_q)$, $q = p^n$, to which 3.1 applies, shows that the first hypothesis in the preceding proposition is necessary.

6. Relation with Alperin’s Fusion Theorem

Our aim here is to mention a connection between $\pi_0\mathcal{X}_p(G)$ and Alperin’s theorem on fusion [1] which has been discovered by Puig [8]. Let $G$ be a finite group of order divisible by $p$. If $H$ is a $p$-subgroup, we let $\mathcal{Y}_p(G)_{>H}$ be the subposet of $\mathcal{Y}_p(G)$ consisting of subgroups $>H$; hence it equals $\mathcal{Y}_p(G)$ when $H = 1$.

**Proposition 6.1.** The poset $\mathcal{Y}_p(G)_{>H}$ is homotopy equivalent to $\mathcal{Y}_p(N_G(H)/H)$.

**Proof.** If $K$ is a $p$-subgroup of $G$ such that $K > H$, one knows that $N_K(H) = N_G(H) \cap K > H$, hence if $i: \mathcal{Y}_p(N_G(H))_{>H} \to \mathcal{Y}_p(G)_{>H}$ denotes the inclusion, we have a map $r$ going the other way given by $r(K) = N_K(H)$. Since $ri=id$ and $ir(K) \leq K$, it follows from 1.3 that $i$ and $r$ are homotopy inverses. But $\mathcal{Y}_p(N_G(H))_{>H}$ is clearly isomorphic to $\mathcal{Y}_p(N_G(H)/H)$, so the proposition follows.

Now let $P$ be a fixed Sylow $p$-subgroup of $G$ and let $\mathcal{K}$ denote the set of subgroups $H$ of $P$ such that (i) $N_P(H) \in \text{Syl}_p(N_G(H))$ and (ii) either $\mathcal{Y}_p(N_G(H)/H)$ is empty or it is disconnected. We note that if $H \in \mathcal{K}$, then $H$ is a “tame intersection” in Alperin’s sense, that is, there exists a $Q \in \text{Syl}_p(G)$ such that $H = P \cap Q$, and $N_G(H)$, as well as $N_P(H)$, are Sylow subgroups of $N_G(H)$. Indeed, by 6.1 one can choose a Sylow group $Q_0$ of $N_G(H)$ not in the same component of $\mathcal{Y}_p(G)_{>H}$ as $P$; then if $Q \in \text{Syl}_p(G)$ contains $Q_0$, one has a tame intersection $H = P \cap Q$. We note also that there are tame intersections $H = P \cap Q$ such that (ii) does not hold, e.g., $H = 1$ in $SL_3(F)$ by 3.1.

Puig’s point is that in Alperin’s analysis of fusion the various tame intersections $P \cap Q_i$ required can all be taken to be in the set $\mathcal{K}$. For example, one has:
Proposition 6.2. If $A, B$ are subsets of $P$ which are conjugate in $G$, then there exist subgroups $H_1, \ldots, H_m$ in $\mathcal{H}$ and elements $y_i$ in $N_G(H_i)$ such that $A^{y_1 \cdots y_m} \subset H_i$ for $i = 1, \ldots, m$ and such that $B = A^{y_1 \cdots y_m}$.

7. LOCAL SYSTEMS AND HOMOLOGY

In this section we collect various facts about the covering spaces and the homology of the simplicial complex associated to a poset. All of the assertions can be generalized without essential change from posets to small categories.

By a local system of sets (resp. abelian groups) on a poset $X$ we mean a functor $F$ from $X$, viewed as a category, to the category of sets (resp. abelian groups) which is morphism inverting, i.e., such that the map $F(x') \to F(x)$ associated to $x' \leq x$ is an isomorphism. The category of local systems of sets on $X$ with natural transformations of functors as morphisms will be denoted Cov$(X)$ because of the following considerations.

Let $E$ be a covering space of $|X|$. One obtains a local system of sets on $X$ by letting $E(x)$ be the fiber of $E$ over $x$ viewed as a vertex of $|X|$, and by associating to $x' \leq x$ the isomorphism obtained by lifting the one-simplex $(x', x)$ to $E$. Conversely if $F$ is a local system on $X$, one can form the poset $X_F$ consisting of pairs $(x, t)$ with $x \in X$ and $t \in F(x)$, with $(x', t') \leq (x, t)$ when $x' \leq x$ and $t'$ corresponds to $t$ under the given isomorphism $F(x') \cong F(x)$. Then $|X_F|$ is a covering space of $|X|$. In this way one obtains an equivalence between the category of covering spaces of $|X|$ and the category Cov$(X)$ of local systems of sets on $X$. (For details, see [5, Appendix I].)

We will use the equivalence between covering spaces and local systems to describe those posets $X$ such that $|X|$ is simply connected, in which case we say $X$ is simply connected. First note that the map $f: X \to pt$ induces a functor

$$\text{Sets} = \text{Cov}(pt) \xrightarrow{f^*} \text{Cov}(X) \quad (7.1)$$

which can be interpreted as associating to a set $S$ the trivial covering space with fiber $S$. It is easily seen that one has the following equivalences:

$$X \neq \emptyset \iff f^* \text{ is faithful},$$
$$X \text{ is connected } \iff f^* \text{ is fully faithful},$$
$$X \text{ is simply connected } \iff f^* \text{ is an equivalence of categories.} \quad (7.2)$$

We note also that $|X|$ is connected iff $X$ is connected as a poset, i.e., any two elements are joined by a sequence whose successive members are related by $\leq$ or $\geq$. 
If $F: X \to Ab$ is a functor from $X$ to abelian groups, we define the homology groups $H_i(X, F)$ of $X$ with coefficients $F$ to be the homology of the complex $C_*(X, F)$ given by

$$C_p(X, F) = \bigoplus_{x_0 < \cdots < x_p} F(x_0)$$

where the direct sum is taken over all $p$-simplices in $X$, with the usual differential $d = \sum (-1)^i d_i$. When $F$ is the constant functor $\mathbb{Z}$, these homology groups coincide with the integral homology of $|X|$, and we write $H_i(X)$ instead of $H_i(X, \mathbb{Z})$. We let $\tilde{H}_i(X)$ denote the reduced integral homology ($= \text{Ker}(H_i(X) \to H_i(pt))$ if $X \neq \emptyset$ and

$$\tilde{H}_i(\emptyset) = \begin{cases} \mathbb{Z} & i = -1 \\ 0 & i \neq -1 \end{cases}.$$

If $f: X \to Y$ is a map of posets and $F: X \to Ab$ is a functor, then we can restrict $F$ to the subposet $f/\gamma$ of $X$ (cf. 1.6) and form the homology $H_\gamma(f/\gamma, F)$. This gives a functor $\gamma \mapsto H_\gamma(f/\gamma, F)$ from $Y$ to Abelian groups. One has a spectral sequence

$$E^2_{pq} = H_p(Y, y \mapsto H_q(f|y, F)) \Rightarrow H_{p+q}(X, F)$$

obtained as follows. One shows that the functors $F \mapsto H_i(X, F)$, $i \geq 0$, are the left-derived functors of the functor "inductive limit over $X."$ Then 7.4 is the Grothendieck spectral sequence calculating the derived functors of the composite functor

$$H_\gamma(X, F) - H_\gamma(Y, y \mapsto H_\gamma(f|y, F)).$$

(For details, see [5, Appendix II].)

We call the poset $X$ $n$-connected when $|X|$ is an $n$-connected space. From topology one has the following criterion:

$$X \text{ is } n\text{-connected} \iff \tilde{H}_i(X) = 0 \quad \text{for } i \leq n \text{ and } |X| \text{ is simply connected if } n \geq 1.$$  

(7.5)

In particular, $(-1)$-connected means nonempty, $0$-connected means connected, and $1$-connected means simply connected.

**Proposition 7.6.** Let $f: X \to Y$ be a map of posets such that $f|y$ is $n$-connected for each $y$ in $Y$. Then $X$ is $n$-connected iff $Y$ is $n$-connected.

**Proof.** The cases $n = -1, 0$ are trivial to check, so we suppose $n \geq 1$. If $E \in \text{Cov}(X)$, we define a functor $f_*E: Y \to \text{Sets}$ by

$$(f_*E)(y) = \lim_{f|y} E.$$
Because \( f/y \) is simply connected, we have that \( E(x) \cong (f\ast E)(y) \) whenever \( x \in f/y \). It follows that \( f\ast E \) is a local system on \( Y \) whose pull-back by \( f \) is isomorphic to \( E \). Thus the pull-back functor \( f\ast : \text{Cov}(Y) \to \text{Cov}(X) \) is an equivalence of categories with inverse functor \( f_! \), so using (7.2) we see \( X \) is simply connected iff \( Y \) is.

Next taking \( F = \mathbb{Z} \) in (7.4) gives a spectral sequence

\[
E^2_{pq} = H^p(Y, y \mapsto H_q(f/y)) \Rightarrow H_{p+q}(X) \tag{7.7}
\]

with \( E^2_{pq} = H^p(Y) \) if \( q = 0 \), and \( E^2_{pq} = 0 \) for \( 0 < q \leq n \). It follows that \( f \) induces isomorphisms \( H_i(X) \cong H_i(Y) \) for \( i \leq n \), so using (7.5) we conclude \( X \) is \( n \)-connected iff \( Y \) is.

We conclude this section with a proof of 1.6, based on the Whitehead theorem which can be formulated for posets as follows.

**Proposition 7.8.** A map of posets \( f : X \to Y \) is a homotopy equivalence iff the following conditions hold.

(a) \( f\ast : \text{Cov}(Y) \to \text{Cov}(X) \) is an equivalence of categories.

(b) The map \( f \) induces an isomorphism \( H_*(X, f\ast L) \cong H_*(Y, L) \) for every local system of abelian groups \( L \) on \( Y \).

By the well-known relation between coverings and fundamental group, condition (a) is equivalent to the requirement that \( f \) induces a bijection between components of \( |X| \) and \( |Y| \) and an isomorphism of the fundamental groups at corresponding points. Hence this proposition follows from the Whitehead theorem asserting that a map of connected CW complexes is a homotopy equivalence iff it induces isomorphisms on fundamental groups and homology with arbitrary twisted coefficients. To prove 1.6 one argues as in the proof of 7.6 and shows that conditions (a) and (b) hold, assuming \( f/y \) is contractible for each \( y \) in \( Y \). If instead \( y\setminus f \) is contractible for all \( y \), then one replaces \( f \) by \( f_{op} : X_{op} \to Y_{op} \) and uses the homeomorphism (1.1).

8. Spherical and Cohen-Macaulay Posets

The concepts of spherical and Cohen-Macaulay posets provide a convenient framework for the results about \( \mathcal{A}_p(G) \) to be presented later. In this section we collect various facts about these posets which will be needed.

Let \( K \) be a finite-dimensional simplicial complex and let \( d \) be its dimension. We call \( K \) \( d \)-spherical, or just spherical, if it is \( (d-1) \)-connected, or equivalently, if it has the homotopy type of a bouquet of \( d \)-spheres. We say that \( K \) is Cohen-Macaulay (CM for short), if it is spherical, and if in addition the link of each \( p \)-simplex in \( K \) is \( (d-p-1) \)-spherical (this condition implies that the
link of each point of $K$ is $(d - 1)$-spherical). A poset will be called spherical or Cohen-Macaulay when the associated simplicial complex has this property.

This definition of Cohen-Macaulay complex is stronger than the customary one [12] where $n$-connectedness is replaced by the vanishing of homology in degrees $\leq n$. Stanley and Baclawski have pointed out that the Edwards construction of a homology 3-sphere double-suspending to a 5-sphere shows that the stronger Cohen-Macaulay property is not homeomorphism invariant. Our goal in this paper is to show certain posets are CM, hence we use the stronger definition so that our theorems are in their best form.

**Examples.**

8.1. If $K$ and $K'$ are spherical simplicial complexes of dimensions $d$ and $d'$, respectively, then it is easily seen that their join $K \ast K'$ is $(d + d' + 1)$-spherical. Consequently by 1.9, if $X$ and $X'$ are spherical posets of dimensions $d$ and $d'$, respectively, then $X \ast X'$ and the homeomorphic poset $CX \times CY - \{ (0, 0) \}$ are $(d + d' + 1)$-spherical. One can prove the same assertions hold with spherical replaced by CM.

8.2. The buildings of Tits [13] are examples of CM simplicial complexes. In particular, the poset $T(V)$ of proper subspaces of a vector space $V$ is a CM poset; this example will be discussed more thoroughly at the end of this section.

8.3. If $X$ is a finite-dimensional poset which is a semimodular lattice, then $X$ is CM. This follows from a theorem of Folkman (cf. [12]) whose proof in a special case is sketched at the end of this section.

We need to introduce some notation. If $x$ is an element of a poset $X$, we let $X_{>x}$ denote the subposet consisting of $x' > x$, and we define $X_{\geq x}, X_{< x}, X_{\leq x}$ in a similar fashion. The **height** of $x$, denoted $h(x)$, is the dimension of the poset $X_{< x}$, that is, the supremum of the lengths of chains having $x$ as largest element.

We recall that the link of a simplex $\sigma$ in a simplicial complex is the subcomplex consisting of those simplices disjoint from $\sigma$ whose union with $\sigma$ is a simplex. Thus the link of the simplex $\sigma = \{x_0 < \cdots < x_p\}$ in $|X|$ is the simplicial complex associated to the subposet of $X$ consisting of elements not in $\sigma$ which can be adjoined to $\sigma$ to form a chain. This subposet we denote $\text{Link}(\sigma, X)$. Clearly one has

$$\text{Link}(\sigma, X) = X_{< x_0} \ast (x_0, x_1) \ast \cdots \ast X_{> x_p}$$  \hspace{1cm} (8.4)$$

where we put $(x', x) = X_{\geq x'} \cap X_{\leq x}$ when $x' < x$.

Next suppose $X$ is finite dimensional. Given $x$ in $X$, we can choose a maximal chain $\sigma = \{x_0 < \cdots < x_p\}$ with $x_p = x$, whence we have $p = h(x)$ and
\[ \text{Link}(\alpha, X) = X_{\geq \alpha}. \] Supposing \( X \) is CM of dimension \( n \), one sees from the definition of CM that \( X_{\geq x} \) is spherical of dimension \( n - h(x) - 1 \). In particular, any maximal element \( x \) has height \( n \), because in this case \( X_{\geq x} \) is empty, hence is of dimension \(-1\).

In a similar way one sees that in a CM poset of dimension \( n \) the following conditions are satisfied:

\[
\begin{align*}
X & \quad \text{is } n\text{-spherical}, \\
X_{\geq x} & \quad \text{is } (n - h(x) - 1)\text{-spherical}, \\
X_{< x} & \quad \text{is } (h(x) - 1)\text{-spherical}, \quad (8.5) \\
(x', x) & \quad \text{is } (h(x) - h(x') - 2)\text{-spherical},
\end{align*}
\]

for all \( x \) and \( x' < x \) in \( X \). Conversely, using (8.1) and (8.4) one sees that a poset satisfying these four conditions is CM. Consequently, if this converse is applied to each of the posets appearing in (8.5), one concludes that in a CM poset of dimension \( n \) the conditions (8.5) hold with spherical replaced by CM.

We summarize this discussion for later reference.

**Proposition 8.6.** A poset is CM of dimension \( n \) if the conditions (8.5) are satisfied. Any CM poset of dimension \( n \) satisfies (8.5) with spherical replaced by CM. In a CM poset of dimension \( n \), any maximal element has height \( n \).

We finish this section with a discussion of the poset \( T(V) \) of proper subspaces in a vector space \( V \) of dimension \( n \). One knows that \( T(V) \) is spherical of dimension \( n - 2 \) either from the theory of buildings, or by the argument of Folkman which goes as follows. One covers \( T(V) \) by the sets \( T(V)_{\geq L} \), where \( L \) ranges over the set \( P \) of lines in \( V \). The finite intersections of members of this covering are either empty or contractible, so one knows \( T(V) \) has the same homotopy type as the nerve of this covering. However, any nonempty subset \( \{L_0, \ldots, L_s\} \) of \( P \) with \( s \leq n - 2 \) is a simplex in this nerve, since \( L_0 + \cdots + L_s < V \), so the nerve contains the \((n - 2)\)-skeleton of the full simplex with vertices in \( P \). It follows that the nerve, hence \( T(V) \) is \((n - 3)\)-connected, so \( T(V) \) is \((n - 2)\)-spherical.

If \( W' < W \) are proper subspaces, then the open interval \((W', W)\) in \( T(V) \) can be identified with \( T(W/W') \). Hence, if \( \sigma = \{W_0 < \cdots < W_s\} \) is a simplex in \( T(V) \), one has an isomorphism

\[
\text{Link}(\sigma, T(V)) = T(W_0) \ast T(W_1/W_0) \ast \cdots \ast T(V/W_s) \quad (8.6)
\]

so this link is spherical of dimension \((n - 2) - s - 1\) by 8.1. Thus \( T(V) \) is CM of dimension \( n - 2 \).
9. SPHERICAL AND COHEN–MACAULAY POSETS (CONTINUED)

In this section we present a theorem which allows one to show certain posets have the properties of being spherical or Cohen–Macaulay.

THEOREM 9.1. Let \( f: X \to Y \) be a map of posets. Assume \( Y \) is \( n \)-spherical, and for each \( y \) in \( Y \) that \( Y_{\geq y} \) is \( (n - h(y) - 1) \)-spherical and \( f/_{y} \) is \( h(y) \)-spherical. Then \( X \) is \( n \)-spherical. Moreover, there is a canonical filtration

\[
0 = F_{n+1} \subset F_n \subset \cdots \subset F_{-1} = \tilde{H}_n(X)
\]

and isomorphisms

\[
F_{-1}/F_0 \cong \tilde{H}_n(Y),
F_{-q}/F_{-q+1} \cong \bigoplus_{h(y)=q} \tilde{H}_{n-q-1}(Y_{\geq y}) \otimes \tilde{H}_q(f/y)
\]

for \( 0 \leq q \leq n \), where the direct sum is taken over the elements of height \( q \) in \( Y \).

Remark. Since the top homology group of a spherical poset is free abelian, \( \tilde{H}_n(X) \) is isomorphic (noncanonically) to the direct sum of the groups (9.2).

Proof. One has

\[
\dim(X) = \sup_{y} \dim(f/y) = \sup_{y} h(y) = \dim(Y) = n
\]

so \( X \) will be \( n \)-spherical once we show it is \( (n - 1) \)-connected. We first compute the homology of \( X \) by the spectral sequence (7.7):

\[
E^{pq}_{pq} = H_p(Y, y \mapsto H_q(f/y)) \Rightarrow H_{p+q}(X).
\]

One has

\[
E^{pq}_{pq} = H_p(Y, y \mapsto \tilde{H}_q(f/y)) \quad \text{for} \quad q > 0,
\]

and for \( q = 0 \) one has a long exact sequence

\[
\rightarrow H_{p+1}(Y) \rightarrow H_p(Y, y \mapsto \tilde{H}_0(f/y)) \rightarrow E^{2}_{pq} \rightarrow H_p(Y) \rightarrow
\]

associated to the short exact sequence

\[
0 \rightarrow \tilde{H}_q(f/y) \rightarrow H_q(f/y) \rightarrow \mathbb{Z} \rightarrow 0.
\]

Since \( f/y \) is assumed to be \( h(y) \)-spherical, the functor \( y \mapsto \tilde{H}_q(f/y) \) vanishes except for \( y \) of height \( q \). We have to calculate the homology of \( Y \) with such a functor as coefficients.

Let \( L \) be an abelian group, and \( y \) a fixed element of \( Y \), and let \( L_{\langle y \rangle} \) denote the functor from \( Y \) to abelian groups sending \( y \) to \( L \) and all other elements
of $Y$ to $0$. Put $U = Y_{>y}$, $V = Y_{>u}$, and let $L_U$ (resp. $L_V$) be the constant functor with value $L$ on $U$ (resp. $V$) extended by 0 to the rest of $Y$. It follows immediately from the definition of homology (7.3) that one has

$$
H_i(Y, L_V) = H_i(V, L)
$$

$$
H_i(Y, L_U) = H_i(U, L) = \begin{cases} L & i = 0 \\ 0 & i \neq 0 \end{cases}
$$

where the last formula comes from the fact that $U$ is contractible as it has a least element. From the long exact sequence associated to the short exact sequence of functors

$$0 \to L_V \to L_U \to L_{(u)} \to 0$$

one gets the formula

$$H_i(Y, L_{(u)}) = \tilde{H}_{i-1}(Y_{>y}, L). \quad (9.5)$$

Because $f/y$ is assumed to be $h(y)$-spherical, the functor $y' \mapsto \tilde{H}_q(f/y')$ is isomorphic to the direct sum of the functors $\tilde{H}_q(f/y)_{(u)}$ as $y$ ranges over the elements of height $q$ in $Y$. Thus one obtains from (9.5) the formula

$$H_p(Y, y \mapsto \tilde{H}_q(f/y)) = \bigoplus_{h(y) = q} H_p-1(Y_{>y}, \tilde{H}_q(f/y)).$$

Using now the hypothesis that $Y_{>y}$ is $(n - h(y) - 1)$-spherical and the fact that the top homology of a spherical complex is free, this may be rewritten

$$H_p(Y, y \mapsto \tilde{H}_q(f/y)) = \bigoplus_{h(y) = q} \tilde{H}_{n-q-1}(Y_{>y}) \otimes H_q(f/y)$$

if $p + q = n$, and =0 if $p + q \neq n$. Combining this with (9.3) and (9.4) and the hypothesis that $Y$ is $n$-spherical, we see $E^2_{p,q} = 0$ for $p + q \neq n$. Thus the spectral sequence degenerates, showing that $\tilde{H}_i(X) = 0$ for $i \neq n$ and also that there exists on $\tilde{H}_n(X)$ a filtration with the quotients required in (9.2).

To finish, we have to show $X$ is simply connected if $n > 2$, for then by (7.5), $X$ is $(n - 1)$-connected. Let $E$ be a local system of sets on $X$, and consider the functor from $Y$ to sets given by

$$(f !E)(y) = \lim_{x \in f/y} E(x).$$

If $h(y) \geq 2$, then $f/y$ is simply connected by hypothesis, hence $E$ restricted to $f/y$ is trivial, i.e., isomorphic to a constant functor, so the canonical maps

$$E(x) \to (f !E)(y), \quad x \in f/y \quad (9.6)$$

are isomorphisms. If $h(y) = 1$, then $Y_{>y}$ is spherical of dimension $n - 2 \geq 0$, hence it is nonempty; choosing a $y' > y$, one has $h(y') \geq 2$ and $f/y \subset f/y'$;
hence $E$ restricted to $f/y$ is trivial and the maps (9.6) are isomorphisms as $f/y$ is connected. Finally, if $h(y) = 0$, then $(f_!E)(y)$ is the disjoint union of the sets $E(x)$ as $x$ ranges over the 0-dimensional set $f/y$. Choose a $y' > y$, and let $F(y)$ be the quotient of the set $(f_!E)(y)$ which is isomorphic to the image of the map $(f_!E)(y) \to (f_!E)(y')$. Because $Y_{>y}$ is connected (it is $(n - 1)$-spherical and $n \geq 2$), this quotient set $F(y)$ is independent of the choice of $y'$. Moreover, we obtain a quotient functor $F$ of $f_!E$ by putting $F(y) = (f_!E)(y)$ for $h(y) \geq 1$. It is clear that one has canonical isomorphisms $E(x) \cong F(y)$ for all $x, y$ with $x \in f/y$, hence $F$ is a local system on $Y$ whose pull-back by $f$ is isomorphic to $E$. Since $Y$ is simply connected, $F$ is trivial, so $E$ is trivial. Thus every covering space of $|X|$ is trivial, and since $X$ is connected by our homology calculation we conclude $X$ is simply connected. This concludes the proof of the theorem.

**Corollary 9.7.** Let $f: X \to Y$ be a map of posets which is strictly increasing, i.e., $x' < x \Rightarrow f(x') < f(x)$. Assume $Y$ is CM of dimension $n$ and that $f/y$ is CM of dimension $h(y)$ for all $y$ in $Y$. Then $X$ is CM of dimension $n$.

**Proof.** We check that the conditions (8.5) are satisfied. Since $Y_{>y}$ is CM of dimension $n - h(y) - 1$ by 8.6, the preceding theorem implies immediately that $X$ is $n$-spherical. Fix an element $x$ in $X$ and put $y = f(x)$. Because $f/y$ is closed in $X$, the height of an element of $f/y$ is the same as its height in $X$. Consequently, as $f/y$ is assumed to be CM, the subposets $X_{<x}$ and $(x', x)$ of $f/y$ are spherical of dimensions $h(x) - 1$ and $h(x) - h(x') - 2$, respectively. It remains to show $X_{>x}$ is $(n - h(x) - 1)$-spherical.

Because $f$ is strictly increasing, it induces a map $f': X_{>x} \to Y_{>y}$; and also $x$ is a maximal element of $f/y$, so $h(x) = h(y)$ by 8.6. We are going to apply the above theorem to $f'$. The poset $Y_{>y}$ is CM of dimension $n' = n - h(y) - 1$, and if $y' \in Y_{>y}$, one has

$$f'(x') = \{ x' \in X_{>x} | f(x') \leq y' \} = (f/y')_{>x}$$

is CM of dimension $h(y') - h(x) - 1 = h(y') - h(y) - 1$ — the height of $y'$ in $Y_{>y}$. Therefore applying the theorem to $f'$ as in the first part of this proof where we showed $X$ is $n$-spherical, we see that $X_{>x}$ is spherical of dimension $n' = n - h(x) - 1$.

Q.E.D.

**10. The CM Property for $\mathcal{A}_p(G)$**

In the rest of this paper we will be interested in groups $G$ with the property that $\mathcal{A}_p(G)$ is a Cohen–Macaulay poset. For example, this is true if $A$ is a $p$-torus, for $\mathcal{A}_p(A)$ can be identified with the cone of the CM poset $T(A)$ of 8.2, where $A$ is regarded as a vector space over $F_p$. 
Proposition 10.1. The poset $\mathcal{P}(G)$ is CM iff $\mathcal{P}(G)_{>B}$ is spherical of dimension $r_p(G) - r_p(B) - 1$ for every $p$-torus $B$ in $G$ (including $B = 1$).

Proof. If $\sigma = \{A_0 < \cdots < A_s\}$ is a simplex in $\mathcal{P}(G)$, one has an isomorphism
\[\text{Link}(\sigma, \mathcal{P}(G)) = T(A_0) * \cdots * T(A_s/A_{s-1})* \mathcal{P}(G)_{>A_s},\]
where all the factors in the join are spherical except possibly the last. Assuming $\mathcal{P}(G)_{>A_s}$ is spherical of dimension $r_p(G) - r_p(A_s) - 1$, one sees from 8.1 that $\text{Link}(\sigma, \mathcal{P}(G))$ is spherical of dimension $(r_p(G) - 1) - s - 1$, proving the sufficiency part. On the other hand, if $B \in \mathcal{P}(G)$, then $\mathcal{P}(G)_{>B} = \text{Link}(\sigma, \mathcal{P}(G))$ where $\sigma$ is the chain of nonidentity subgroups in a composition series for $B$, hence the necessity part is clear.

Remark 10.2. If $\mathcal{P}(G)$ is CM, then all maximal $p$-tori in $G$ have rank $r_p(G)$ by the above (or 8.6). Consequently, groups such that $\mathcal{P}(G)$ is CM are somewhat special.

Proposition 10.3. If $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ are CM, then so is $\mathcal{P}(G_1 \times G_2)$.

Proof. Let $G = G_1 \times G_2$, let $pr_i : G \to G_i$ be the projections, and let $B$ be a $p$-torus of $G$. We wish to show $\mathcal{P}(G)_{>B}$ is spherical of dimension $r_p(G) - r_p(B) - 1$. For any $p$-torus $A$ in $G$ we have $A \subseteq pr_1(A) \times pr_2(A)$, hence any maximal $p$-torus in $G$ is of the form $A_1 \times A_2$ with $A_i$ a maximal $p$-torus of $G_i$. It follows that one has $r_p(G) = r_p(G_1) + r_p(G_2)$ and $\dim \mathcal{P}(G)_{>B} = r_p(G) - r_p(B) - 1$, hence it suffices to show $\mathcal{P}(G)_{>B}$ is $(r_p(G) - r_p(B) - 2)$-connected.

Put $B_i = pr_i(B)$, $T_i = \mathcal{P}(G_i)_{>B_i}$, and let $T$ be the subset of $\mathcal{P}(G)_{>B}$ consisting of $p$-tori of the form $A_1 \times A_2$ with $A_i \subseteq G_i$. Then $T$ is isomorphic to the poset $CT_1 \times CT_2$ if $B < B_1 \times B_2$, and to $CT_1 \times CT_2 - \{(0,0)\}$ if $B = B_1 \times B_2$. In the former case $T$ is contractible, and in the latter $T$ is homeomorphic to $T_1 * T_2$ by 1.9, which is $(r_p(G) - r_p(B) - 1)$-spherical by 8.1, because $T_i$ is $(r_p(G_i) - r_p(B_i) - 1)$-spherical by hypothesis and 10.1. Thus $T$ is $(r_p(G) - r_p(B) - 2)$-connected. On the other hand, the map $A \mapsto pr_1(A) \times pr_2(A)$ from $\mathcal{P}(G)_{>B}$ to $T$ is a homotopy equivalence (cf. proof of 2.6), so the proof is complete.

Example 10.4. Let $p$ be an odd prime and let $G$ be an extension of a $p$-torus $V$ by a central subgroup $H$ which is cyclic of order $p$. We claim that for such a group $\mathcal{P}(G)$ is CM. To see this, we recall that associated to such an extension is a homomorphism $u : V \to H$ and a skew-symmetric bilinear form $f : V \times V \to H$ obtained by lifting back to $G$ and taking the $p$th power and commutator, respectively. Thus a subgroup of $G$ is a $p$-torus iff its image is killed by $u$ and $f$.  

The claim reduces easily to the case where $u$ is zero and $f$ is nondegenerate. In this case $\mathcal{A}_p(G,H)$ is isomorphic to the poset of nonzero subspaces of $V$ isotropic for $f$; this is CM of dimension $\frac{1}{2} \dim V - 1$, because the associated simplicial complex is the building for the symplectic group of $V$. Hence $\mathcal{A}_p(G,B)$ is spherical of the right dimension for any $B$ containing $H$, and if $B$ does not contain $H$, then it is conically contractible: $A \leq AH \geq BH$.

11. Extensions by Certain Solvable Groups

In this section we apply the results of Sections 9 about CM posets to prove a key theorem about the behavior of the CM property under extensions by a solvable subgroup which is uniquely $p$-divisible in a certain sense.

Let $A$ be a $p$-torus, and let $P(A)$ be the set of hyperplanes in $A$, i.e., subgroups $B$ such that $A/B$ is cyclic of order $p$. We consider $A$-modules $M$ which are uniquely $p$-divisible as abelian groups, that is, modules over $\mathbb{Z}[p^{-1}]$. On the category of such modules the functor of taking invariants: $M \mapsto M^A$ is exact. For each $B$ in $P(A)$ one has a canonical $A$-module decomposition

$$M^B = M^A \oplus M_{(B)}$$

where $M_{(B)}$ is the subgroup generated by $am - m$ with $a \in A$ and $m \in M^B$. Since $M_{(B)}$ is an $A/B$-module without nonzero invariants, the cyclic group $A/B$ acts freely on $M_{(B)} - \{0\}$. Finally, we claim there is an $A$-module decomposition

$$M = M^A \oplus \bigoplus_{B \in P(A)} M_{(B)} \ . \quad (11.1)$$

To see this, first note that the assertion is true when $M$ is an irreducible representation of $A$ over one of the residue fields of $\mathbb{Z}[p^{-1}]$. This is because an irreducible representation is given by a character of $A$ into roots of unity, and the kernel of a character is either a hyperplane in $A$ or all of $A$. Second, note that both sides of (11.1) are exact functors of $M$ which commute with filtered inductive limits. Hence, starting from irreducible modules, one sees (11.1) holds for torsion uniquely $p$-divisible $A$-modules, as well as uniquely divisible $A$-modules. But then, as (11.1) holds for $M \otimes \mathbb{Q}$, the torsion submodule of $M$, and the torsion module $M \otimes \mathbb{Q}/M$, one sees by exactness that it holds for $M$.

We can now prove the following.

**Theorem 11.2.** Let $G$ be a group having a chain of normal subgroups $H = H_0 \supset H_1 \supset \cdots \supset H_s = 1$ such that $G/H$ is a $p$-torus of rank $r$ and $H_i/H_{i+1}$ is a uniquely $p$-divisible abelian group for $0 \leq i < s$. Then
(i) $\mathcal{A}_p(G)$ is CM of dimension $r - 1$,

(ii) if $G$ contains no central elements of order $p$, then

$$\bar{H}_{r-1}(\mathcal{A}_p(G)) \neq 0.$$ 

Before beginning the proof we make some remarks. The first is that the group $H$ has no elements of order $p$; this follows from the fact that $H/H_{t+1}$ has no elements of order $p$. Second, the extension $G$ of $G/H$ by $H$ splits, i.e., $G$ is a semidirect product $AH$, where $A$ is a $p$-torus of rank $r$. This is because the obstruction to lifting a $p$-torus $A \subset G/H$ to $G/H_{t+1}$ lies in the cohomology group $H^2(A, H/H_{t+1})$, which is zero as $H/H_{t+1}$ is uniquely $p$-divisible. Finally we remark that if the element $ah$ of $G - AH$ is in the center and has order $p$, then $ah = a^{-1}(ah)h = ha$, so $h^p = a^p h^p = (ah)^p = 1$, hence $h = 1$. Consequently, if we let $pZ(G)$ be the subgroup of central elements of order dividing $p$, we have

$$pZ(G) = \text{the centralizer of } H \text{ in } A. \quad (11.3)$$

In particular, the hypothesis in (ii) above is equivalent to requiring that $A$ acts faithfully on $H$.

We prove the theorem by induction on $s$ and begin with the induction step. Supposing $s > 1$, put $G' = G/H_s$ and let $\pi: G \to G'$ be the canonical map. One has $A \simeq \pi A$ for any $p$-torus in $G$, hence $\pi$ induces a map of poset $f: \mathcal{A}_p(G) \to \mathcal{A}_p(G')$ to which we apply 9.7 with $n = r - 1$. By the induction hypothesis $\mathcal{A}_p(G')$ is CM of dimension $r - 1$. One has

$$f/B = \{A \in \mathcal{A}_p(G) \mid \pi A \subset B\} = \mathcal{A}_p(\pi^{-1}B)$$

and $\pi^{-1}B$ is an extension of $B$ by $H_s$; as $s > 1$, the induction hypothesis shows the theorem holds for the group $\pi^{-1}B$. Hence $f/B$ is CM of dimension $r_p(G) - 1$ which is the height of $B$ in $\mathcal{A}_p(G')$. Thus from 9.7 we obtain the assertion (i) for $G$.

Put $B = pZ(G')$ and $t = r_p(B)$. Assuming $pZ(G) = 1$, we are going to prove

(a) $pZ(\pi^{-1}B) = 1$,

(b) $\bar{H}_{r-1}(\mathcal{A}_p(G')_{>B}) \neq 0$.

Using assertion (ii) for the group $\pi^{-1}B$, we see that (a) implies $\bar{H}_{r-1}(f/B) \neq 0$. Referring to the second part of Theorem 9.1, one sees that this combined with (b) implies $\bar{H}_{r-1}(\mathcal{A}_p(G)) \neq 0$, proving assertion (ii) for $G$.

Put $A = pZ(\pi^{-1}B)$. As $H_s$ is abelian and $B$ is central in $G'$, conjugation gives a homomorphism from $A$ to the group $T$ of automorphisms of $G$ inducing the identity on $G'$ and on $H_s$. The group $T$ is a uniquely $p$-divisible abelian group (it is isomorphic to the group of crossed homomorphisms from $G'$ to $H_s$), hence this homomorphism is trivial. Thus $A \subset pZ(G) = 1$, which proves (a).
By the remarks made at the beginning of the proof, \( G' \) is a semidirect product \( A'H' \), where \( A \) is a \( p \)-torus of rank \( r \), and \( H' = H/H_2 \). By (11.3), \( B \) is the subgroup of \( A \) centralizing \( H' \), hence in the semidirect product group \( G'/B = (A/B)H' \) one has no central elements of order \( p \), because \( A/B \) acts faithfully on \( H' \). But one has \( \mathcal{A}_p(G')_{\geq B} = \mathcal{A}_p(G'/B) \), and on the other hand, the theorem holds for the group \( G'/B \) by the induction hypothesis. Thus the assertion (ii) of the theorem for \( G'/B \) yields (b).

At this point the inductive step in the proof has been established, so it remains to treat the case \( s = 1 \), i.e., when \( G \) is a semidirect product \( A'H \), with \( A \) a \( p \)-torus of rank \( r \) and \( H \) a uniquely \( p \)-divisible abelian normal subgroup. Using the decomposition (11.1), one sees there is a chain of \( A \)-submodules \( H = H_0 \supset \cdots \supset H_s = 1 \) such that for each \( i = 0, \ldots, s - 1 \) the quotient \( H_i/H_{i+1} \) is a uniquely \( p \)-divisible \( A \)-module on which either \( A \) acts trivially, or else, one has \( H_i/H_{i+1} > 1 \) and there is a hyperplane \( A_0 \) in \( A \) such that \( A/A_0 \) acts freely on the complement of the identity in this quotient. We again argue by induction on \( s \). The inductive step is completely identical to the one just given, so it remains to check the case \( s = 1 \). There are two subcases:

(1) The group \( A \) acts trivially on \( H \), whence \( G = A \times H \) and \( \mathcal{A}_p(G) = \mathcal{A}_p(A) \) is CM of dimension \( r - 1 \), because \( \mathcal{A}_p(A)_{\geq B} \) is contractible if \( B < A \) and empty if \( B = A \). Thus (i) holds. If \( G \) has no central elements of order \( p \), then \( r = 0 \), so \( \mathcal{A}_p(G) \) is empty and \( \tilde{H}_{-1}(\mathcal{A}_p(G)) = \mathbb{Z} \), proving (ii).

(2) The group \( H > 1 \) and \( A/A_0 \) acts freely on \( H - \{1\} \) for some hyperplane \( A_0 \). If we choose a cyclic subgroup \( A_1 \) in \( A \) complementary to \( A_0 \), then \( G = (A_1H) \times A_0 \). Since all \( p \)-tori in \( A_1H \) have rank \( \leq 1 \), it is clear that \( \mathcal{A}_p(A_1H) \) is CM of dimension \( 0 \). As \( \mathcal{A}_p(A_0) \) is CM of dimension \( r - 2 \), one sees \( \mathcal{A}_p(G) \) is CM of dimension \( r - 1 \) by 10.3, proving (i). On the other hand, if \( G \) has no central elements of order \( p \), then one has \( A_0 = 1 \), and \( \mathcal{A}_p(G) \) is zero dimensional with more than one element, because \( A \) is not normal in \( G \). Hence \( \tilde{H}_0(\mathcal{A}_p(G)) \neq 0 \), proving (ii). This finishes the proof of the theorem.

**Corollary 11.4.** Let \( G \) be a group having a chain of normal subgroups \( H = H_0 \supset H_1 \supset \cdots \supset H_s = 0 \) such that \( H_i/H_{i+1} \) is a uniquely \( p \)-divisible abelian group for \( 0 \leq i < s \). If \( \mathcal{A}_p(G/H) \) is CM, then \( \mathcal{A}_p(G) \) is CM of the same dimension.

This follows from 9.7 applied to the map \( f: \mathcal{A}_p(G) \to \mathcal{A}_p(G/H) \) induced by the canonical homomorphism \( n: G \to G/H \). One has \( f/B = \mathcal{A}_p(n^{-1}B) \) which is CM of dimension \( r_p(B) - 1 \) by the theorem.

**12. Applications**

We now give some applications of the preceding results. Note that the hypotheses of 11.2 and 11.4 imply that the group \( H \) is solvable. In the converse direction we have that if \( H \) is a solvable \( p' \)-group (finite of order prime to \( p \))
or a nilpotent uniquely $p$-divisible group, then the derived series and lower central series, respectively, give the chain of normal subgroups required in these results.

Our first application will be to finite solvable groups.

**Theorem 12.1.** Let $G$ be a finite solvable group having no nontrivial normal $p$-subgroup. If $G$ has a maximal $p$-torus of rank $r$, then $H_{r-1}(\mathcal{A}_p(G)) \neq 0$.

**Proof.** Let $A$ be a maximal $p$-torus in $G$ of rank $r$, and let $H$ be the largest normal $p'$-subgroup of $G$. By the Hall-Higman Lemma 1.2.3, our hypotheses imply that $C_G(H) \subseteq H$, hence $A$ acts faithfully on $H$, and so $H_{r-1}(Y) \neq 0$ by 11.2, where $Y = \mathcal{A}_p(AH)$. Put $X = \mathcal{A}_p(G)$ and let $Z$ be the closed subset of $X$ consisting of all $p$-tori contained in some $p$-torus of $G$ not contained in $AH$. We next show that any $B \in Y \cap Z$ has rank $< r$. In effect, one has $x^{-1}Bx \subseteq A$, as $A$ is a Sylow $p$-subgroup of $AH$, and one has $B < B_1$ for some $p$-torus $B_1$ not in $AH$. Thus $x^{-1}Bx < A$, otherwise we contradict maximality of $A$.

It follows that $Y \cap Z$ has dimension $< r - 1$. Since

$$|X| = |Y| \cup |Z|,$$

$$|Y| \cap |Z| = |Y \cap Z|$$

one has a Mayer–Vietoris sequence

$$\tilde{H}_{r-1}(Y \cap Z) \rightarrow \tilde{H}_{r-1}(Y) \oplus \tilde{H}_{r-1}(Z) \rightarrow \tilde{H}_{r-1}(X)$$

$$(=0) \quad (\neq 0)$$

which shows that $\tilde{H}_{r-1}(X) \neq 0$, as was to be proved.

**Corollary 12.2.** If $G$ is a finite solvable group having no nonidentity normal $p$-subgroup, then $\tilde{H}_d(\mathcal{A}_p(G)) \neq 0$, where $d = r_p(G) - 1 = \dim \mathcal{A}_p(G)$.

This is clear. Note that it shows the conjecture 2.9 is true for solvable groups.

**Problem 12.3.** Show the conclusions of Theorem 11.2 hold for a semidirect product $AH$, where $A$ is a $p$-torus of rank $r$, and $H$ is a normal $p'$-subgroup. If this is true, then 12.1 and 12.2 remain valid when solvable is replaced by $p$-solvable.

Here is another application of the theory developed so far.

**Theorem 12.4.** Let $F$ be a field of characteristic $\neq p$ containing a primitive $p$th root of unity. Then $\mathcal{A}_p(GL_n(F))$ is CM of dimension $n - 1$.

**Proof.** We use induction on $n$, starting from $n = 1$, where the assertion follows from the fact that $F^*$ has a unique cyclic subgroup of order $p$. Put
$G = GL_n(F)$ and let $A$ be a $p$-torus in $G$ of rank $r$. We have to show $\mathcal{A}_p(G)_{>A}$ is spherical of dimension $n - r - 1$. If we decompose $F^n$ according to the characters of $A$

\[ F^n = \bigoplus V_x \tag{12.5} \]

the centralizer of $A$ is

\[ C_G(A) = \prod \text{Aut}(V_x). \]

If the decomposition (12.5) is nontrivial, then the induction hypothesis together with 10.3 shows that $\mathcal{A}_p(C_G(A))$ is CM of dimension $n - 1$, hence

\[ \mathcal{A}_p(G)_{>A} = \mathcal{A}_p(C_G(A))_{>A} \]

is $(n - r - 1)$-spherical. Thus we have to treat the case where $V_x = F^n$ for some $x$, in other words, where $A$ is contained in the group $C$ of scalar matrices with a $p$th root of unity on the diagonal. If $A = 1$, then $\mathcal{A}_p(G)_{>A}$ is $(n - 1)$-spherical, because it has dimension $n - 1$ and it is conically contractible: $B \leq BC \geq C$. It remains to show that $X = \mathcal{A}_p(G)_{>C}$ is $(n - 2)$-spherical. Since $X$ has dimension $n - 2$, we have to show it is $(n - 3)$-connected.

Let $Y$ be the poset of simplices in the building $|T(F^n)|$ belonging to $G$; then $Y$ is $(n - 2)$-spherical as $|Y| = |T(F^n)|$.

**Lemma 1.** If $G_y$ is the stabilizer of $y \in Y$, then $\mathcal{A}_p(G_y)$ is CM of dimension $n - 1$.

Up to conjugacy $G_y$ is one of the standard "parabolics" in $G$:

\[
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & *
\end{pmatrix}
\]

If $H$ is the "unipotent radical" of $G_y$, i.e., the subgroup with identity matrices in the diagonal blocks, then the column filtration of $H$ gives a chain as required in 11.4, because the quotients $H_i/H_{i+1}$ are vector spaces over $F$, hence uniquely $p$-divisible. Thus 11.4 tells us $\mathcal{A}_p(G_y)$ is CM of dimension $n - 1$ provided $\mathcal{A}_p(G_y/H)$ is. However, $G_y/H$ is a product of lower dimensional general linear groups, so this follows from the induction hypothesis and 10.3, proving the lemma.

**Lemma 2.** If $A$ is a $p$-torus in $G$, then $Y^A$ is $(n - 2)$-spherical.

One has

\[ |Y^A| = |Y|^A = |T(F^n)^A| = |T(F^n)|^A \]
where $T(F^n)^A$ is the poset of proper $A$-invariant subspaces of $F^n$. But the poset of all $A$-invariant subspaces is a modular lattice of dimension $n$, so $T(F^n)^A$ is $(n - 2)$-spherical by 8.3, proving the lemma.

Now let $Z$ be the closed subset of $X \times Y$ consisting of $(A, y)$ such that $y \in Y^A$, or equivalently, $A \subseteq G_y$. From the above lemmas we see that $Z_A = Y^A$ and $Z_y = \mathcal{A}(G_y) > C$ are $(n - 3)$-connected. We apply 7.4 to the map $p_2^0: Z_{op} \rightarrow Y_{op}$, using that $p_2^0|_Z = (y| p_2^0)$ is homotopy equivalent to $Z_y^{op}$ (see proof of 1.7). It follows that $Z$ is $(n - 3)$-connected, because we know $Y$ is. Similarly, by considering $p_1: Z \rightarrow X$, we conclude $X$ is $(n - 3)$-connected, which finishes the proof.

REFERENCES