Homology and Shellability of Matroids and Geometric Lattices

(A chapter for "Matroid Applications", ed. N. White)

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CONTENT:

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7.1. Introduction

With a finite matroid M are associated several simplicial complexes which are interrelated in an appealing way. They carry some of the significant invariants of M as face numbers and Betti numbers, and give rise to useful algebraic structures. In this chapter we will study three such complexes: (1) The matroid complex $IN(M)$ of independent subsets, (2) the broken circuit complex $BC_\omega(M)$ relative to an ordering ω of the ground set, and (3) the order complex $\Delta(\bar{L})$ of chains in the associated geometric lattice L.

To systematize our approach to the combinatorial and homological properties of these complexes we utilize the notion of shellability. A complex is said to be shellable if its maximal faces are equicardinal and can be arranged in a certain order which is favorable for induction arguments. Shellability was established for matroid and broken circuit complexes by Provan (1977) and for order complexes of geometric lattices by Björner (1980a).One key property of a shellable complex Δ which we bring into play is the existence of a polynomial $h_\Delta(x)$ with nonnegative integer coefficients that encodes the basic combinatorial invariants of Δ. The coefficients of $h_\Delta(1+\lambda)$ are the face numbers of Δ and $h_\Delta(0)$ is the top Betti number of Δ, all other Betti numbers being zero. Since each coefficient in $h_\Delta(x)$ has an interpretation as counting certain of the maximal faces of Δ, the determination of the homology of a shellable complex becomes a purely combinatorial task once the basic theory of such complexes has been established.

The simplicial complex which most naturally comes to mind in connection with a matroid M is the collection $IN(M)$ of independent sets in M. While exploring the shellability of such complexes we are naturally led to the concepts of internal and external activity in a basis of M, and from there to the consideration of a two–variable generating function $T_M(x,y)$, the Tutte polynomial, such that $T_M(x,1)$ and $T_M(1,y)$ are the shelling polynomials of $IN(M)$ and $IN(M^*)$ respectively (M^* is the orthogonal matroid). As an application, several matroid inequalities are derived.

The broken circuit complex $BC_\omega(M)$ of $M = M(S)$ relative to an ordering ω of S is the collection of those subsets of S which do not contain any broken circuit, that is, circuit with deleted first element. This notion was developed by Whitney (1932), Rota (1964), Wilf (1976) and Brylawski (1977a), originally for enumerative purposes. The broken circuit complex carries the "chromatic" properties of M: the shelling polynomial of $BC_\omega(M)$ equals $T_M(x,0)$, the face numbers are the Whitney numbers of the first kind and $BC_\omega(M)$ is a cone over a related complex whose top Betti number is $\beta(M)$, the beta invariant of M.

The homology of geometric lattice complexes $\Delta(\bar{L})$ was determined in pioneering work of Folkman (1966), and has had a significant role since then. On the one hand, Folkman's vanishing theorem made geometric lattices one of the motivating examples for the theory of Cohen–Macaulay posets (Baclawski, 1980; Björner, Garsia and Stanley, 1982). On the other hand, Orlik and Solomon (1980) showed that the singular cohomology ring of the complement of a complex arrangement of hyperplanes can be described entirely in terms of the order homology of the geometric lattice of intersections. Hence, in these connections (and others, such as in Gel'fand and Zelevinskii, 1986) geometric lattice homology is related to interesting applications of matroids within mathematics.

In connection with the homology of matroid complexes $IN(M)$ and geometric lattice complexes $\Delta(\bar{L})$, an interesting role is played by broken circuit complexes. Namely, $BC_\omega(M^*)$ induces cycles that form a characteristic-free basis for the homology of $IN(M)$, and $BC_\omega(M)$ similarly determines a basis for the homology of $\Delta(\bar{L})$. These are together with the Orlik–Solomon algebra examples of a certain universality of the broken circuit idea for constructing bases for algebraic objects associated to matroids and geometric lattices.

This chapter aims to give a unified and concise, yet gentle, introduction to the topics that have been outlined. A minimum of prerequisites will be assumed. Sections 7.2 – 7.6 are entirely combinatorial; all algebraic aspects have been deferred to the last four sections. A simple presentation of the relevant parts of simplicial homology in Section 7.7 makes the chapter essentially self-contained. Only the most basic ideas are developed in the text, additional results and ramifications appear among the exercises. Section 7.11 contains all references to original sources and related comments.

7.2. Shellable complexes

We begin by recalling the fundamental definitions. A *simplicial complex* (or just *complex*) Δ is a collection of subsets of a finite set V such that (*i*) if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$ and (*ii*) if $v \in V$ then $\{v\} \in \Delta$. Throughout this chapter we will assume that complexes are nonvoid. Note that $\Delta \neq \emptyset$ implies $\emptyset \in \Delta$. The elements of V are called *vertices* and the members of Δ are called *simplices* or *faces*. A face which is not properly contained in any other face is called a *facet*. The *dimension* of a face $F \in \Delta$ is one less than its cardinality, and the *dimension* of the complex is the maximal dimension of a face. That is, $\dim F = |F| - 1$ and $\dim \Delta = \max\{\dim F | F \in \Delta\}$. A complex is said to be *pure* if all its facets are equicardinal.

For a simplicial complex Δ let f_k denote the number of faces of cardinality k. Thus $f_0 = 1, f_1 = |V|$ and $f_k = 0$ for $k > r = \dim \Delta + 1$.

The convention in the literature is to let f_k denote the number of faces of *dimension* k, but from a combinatorial point of view, and particularly for the purposes of this chapter, our definition has definite advantages.

It is convenient to express the *face numbers* f_k by their generating function, the *face enumerator*

$$f_\Delta(\lambda) = \lambda^r + f_1\lambda^{r-1} + \ldots + f_r = \sum_i^r f_i\lambda^{r-i}. \tag{7.1}$$

The *Euler characteristic* of Δ is $\chi(\Delta) = -1 + f_1 - f_2 + \ldots = (-1)^{r-1} f_\Delta(-1)$. (A topologist would call this the "reduced" Euler characteristic, it is one less than the usual topological Euler characteristic.) A complex Δ for which every facet contains a certain vertex v is called a *cone* with *apex* v. Since the number of even faces must equal the number of odd faces (there is a pairing with respect to containment of v), it follows that

$$\chi(\Delta) = 0, \text{ if } \Delta \text{ is a cone .} \tag{7.2}$$

7.2.1. **Example.** Let Δ be the 2–dimensional simplicial complex of Figure 1, having facets $A = \{a,b,c\}$, $B = \{a,b,d\}$, $C = \{a,c,d\}$, $D = \{b,d,e\}$, $E = \{c,d,e\}$ and $F = \{b,c,d\}$. The face numbers are $f_0 = 1$, $f_1 = 5$, $f_2 = 9$ and $f_3 = 6$, hence $\chi(\Delta) = -1 + 5 - 9 + 6 = 1$.

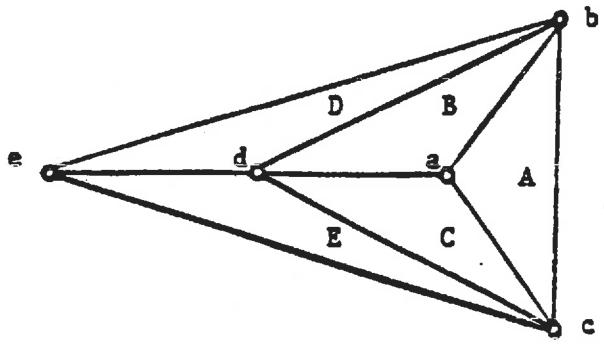


Figure 1

Let Δ be a pure simplicial complex. A *shelling* of Δ is a linear order of the facets of Δ such that each facet meets the complex generated by its predecessors in a nonvoid union of maximal proper faces. In other words, the linear order $F_1, F_2, \ldots, F_t$ of the facets of Δ is a shelling if and only if

(7.3) *for each pair F_i, F_j of facets such that $1 \leq i < j \leq t$ there is a facet F_k satisfying $1 \leq k < j$ and an element $x \in F_j$ such that $F_i \cap F_j \subseteq F_k \cap F_j = F_j - x$.*

A complex is said to be *shellable* if it is pure and admits a shelling. It is easy to see that every 0-dimensional complex is shellable, while a 1-dimensional complex (a simple graph) is shellable if and only if it is connected. The intuitive idea with a shelling is that of building the pure d-dimensional complex Δ stepwise by introducing one facet at a time and attaching it onto the complex already constructed in such a nice way that the intersection is topologically a $(d-1)$–ball or a $(d-1)$–sphere.

As an example, consider the complex Δ in Figure 1. It is easy to verify that an arbitrary permutation of A, B and C followed by an arbitrary permutation of D, E and F gives a shelling of Δ, whereas linear orders of the facets which begin with $A, B, E, \ldots$, or with $D, E, C, A, \ldots$ are not shellings.

For the remainder of this section let us keep the following notation fixed. Δ is an $(r-1)$–dimensional shellable complex, and $F_1, F_2, \ldots, F_t$ are the facets of Δ listed in a shelling order. For $i = 1, 2, \ldots, t$ let $\Delta_i = \{G \in \Delta | G \subseteq F_k \text{ for some } k \leq i\}$, that is, Δ_i is the subcomplex of Δ generated by the i first facets. Also, for $i = 1, 2, \ldots, t$ let $\mathcal{R}(F_i) = \{x \in F_i : F_i - x \in \Delta_{i-1}\}$, called the *restriction* of F_i induced by the shelling. Thus $\mathcal{R}(F_i) = \emptyset$ if and only if $i = 1$ and $\mathcal{R}(F_i) = F_i$ if and only if all proper subsets of F_i are contained in Δ_{i-1}. In the following proposition we consider Δ to be partially ordered by set inclusion of the faces, so that we may speak of Boolean intervals $[G_1, G_2] = \{G \in \Delta : G_1 \subseteq G \subseteq G_2\}$ as subsets of Δ.

7.2.2. **Proposition.** The intervals $[\mathcal{R}(F_i), F_i], i = 1, 2, \ldots, t$, partition the shellable complex Δ.

Proof. The sequence $F_1, F_2, \ldots, F_i$ of facets of Δ_i is a shelling of Δ_i, so inductively it will suffice to show that (i) $\Delta_{t-1} \cup [\mathcal{R}(F_t), F_t] = \Delta$ and (ii) $\Delta_{t-1} \cap [\mathcal{R}(F_t), F_t] = \emptyset$. Let $G \in \Delta - \Delta_{t-1}$. Then $G \subseteq F_t$. If $x \notin G$ for some $x \in \mathcal{R}(F_t)$ then $G \subseteq F_t - x \in \Delta_{t-1}$, which contradicts $G \notin \Delta_{t-1}$. Hence, $\mathcal{R}(F_t) \subseteq G$, and (i) is done. (ii) is equivalent to $\mathcal{R}(F_t) \notin \Delta_{t-1}$. If $\mathcal{R}(F_t) \in \Delta_{t-1}$, then $\mathcal{R}(F_t) \subseteq F_i \cap F_t$ for some $i, 1 \leq i < t$, and so by definition (7.3) $\mathcal{R}(F_t) \subseteq F_i \cap F_t \subseteq F_k \cap F_t = F_t - x$ for some $k, 1 \leq k < t$, and $x \in F_t$.

But $F_t - x \subseteq F_k \in \Delta_{t-1}$ entails that $x \in \mathcal{R}(F_t)$, which contradicts $\mathcal{R}(F_t) \subseteq F_t - x$. So, $\mathcal{R}(F_t) \notin \Delta_{t-1}$. □

The preceding result shows that when the facet F_i (and all its subfaces) is added to the complex Δ_{i-1} during the shelling process, then $\mathcal{R}(F_i)$ is the *unique* minimal face of F_i which is "new" in Δ_i, that is, which lies in $\Delta_i - \Delta_{i-1}$.

With the shellable complex Δ we shall associate the *shelling polynomial* $h_\Delta(x)$, defined by

$$h_\Delta(x) = \sum_{i=1}^{t} x^{|F_i - \mathcal{R}(F_i)|}. \tag{7.4}$$

7.2.3. **Proposition.** Let Δ be a shellable complex with shelling polynomial $h_\Delta(x)$ and face enumerator $f_\Delta(\lambda)$. Then

$$h_\Delta(1+\lambda) = f_\Delta(\lambda).$$

Hence, the polynomial $h_\Delta(x)$ is independent of shelling order.

Proof.

$$h_\Delta(1+\lambda) = \sum_{i=1}^{t}(1+\lambda)^{|F_i-\mathcal{R}(F_i)|} = \sum_{i=1}^{t}\sum_{k=0}^{r}\binom{|F_i - \mathcal{R}(F_i)|}{k}\lambda^k = \sum_{k=o}^{r}\left(\sum_{i=1}^{t}\binom{|F_i - \mathcal{R}(F_i)|}{k}\right)\lambda^k,$$

and by Proposition 7.2.2 $f_{r-k} = \sum_{i=1}^{t}\binom{|F_i - \mathcal{R}(F_i)|}{k}$. □

We see from the above that $h_\Delta(1)$ equals the number of facets and $h_\Delta(2)$ equals the number of faces of Δ. More important is that $h_\Delta(0) = f_\Delta(-1) = (-1)^{r-1}\chi(\Delta)$. Directly from the definition we get, however, that $h_\Delta(0)$ equals the number of facets F such that $\mathcal{R}(F) = F$.

7.2.4. **Corollary.** $(-1)^{r-1}\chi(\Delta)$ equals the number of facets F such that $\mathcal{R}(F) = F$. □

To illustrate these ideas, let us for the complex Δ of Example 7.2.1 choose the shelling A, B, C, D, E, F. Then $\mathcal{R}(A) = \emptyset$, $\mathcal{R}(B) = \{d\}$, $\mathcal{R}(C) = \{c, d\}$, $\mathcal{R}(D) = \{e\}$, $\mathcal{R}(E) = \{c, e\}$, $\mathcal{R}(F) = F$. Hence, the shelling polynomial is

$$h_\Delta(x) = x^3 + x^2 + x + x^2 + x + 1 = x^3 + 2x^2 + 2x + 1.$$

We can now check that $h_\Delta(2) = 21$ is the total number of faces, $h_\Delta(1) = 6$ is the number of facets and $h_\Delta(0) = 1$ equals the Euler characteristic $\chi(\Delta)$. In fact, of course, $h_\Delta(1 + \lambda) = \lambda^3 + 5\lambda^2 + 9\lambda + 6 = f_\Delta(\lambda)$.

Let $f_\Delta(\lambda) = f_0\lambda^r + f_1\lambda^{r-1} + \ldots + f_r$ and $h_\Delta(x) = h_0x^r + h_1x^{r-1} + \ldots + h_r$ be the face enumerator and the shelling polynomial of Δ. The two number sequences $(f_0, f_1, \ldots, f_r)$ and $(h_0, h_1, \ldots, h_r)$, called the *f–vector* and *h–vector* of Δ, respectively, are intimately related.

By comparing coefficients in the relation $f_\Delta(\lambda) = h_\Delta(1 + \lambda)$ (Proposition 7.2.3) we get

$$f_k = \sum_{i=0}^{r} h_i \binom{r-i}{k-i}, k = 0, 1, \ldots, r. \tag{7.5}$$

Similarly, the relation $h_\Delta(x) = f_\Delta(x-1)$ implies the inverse formula

$$(7.6) \qquad h_k = \sum_{i=0}^{r} (-1)^{i+k} f_i \binom{r-i}{k-i}, k = 0, 1, \ldots, r.$$

We deduce that $h_0 = f_0 = 1$ and $h_1 = f_1 - r$. This also follows directly from the definition (7.4) of $h_\Delta(x)$, which can be restated:

$$(7.7) \qquad h_k = \text{ card } \{ \text{ facets } F \text{ such that } |\mathcal{R}(F)| = k\}, 0 \le k \le r.$$

From this we have that $h_k \ge 0$ for $0 \le k \le r$.

We shall now use the correlation between f–vectors and h–vectors to uncover some enumerative facts about shellable complexes. These will be used only in Section 7.5.

7.2.5. **Proposition.** Let Δ be an $(r-1)$–dimensional shellable complex on the vertex set $V, |V| = v > r$. Then

(i) $f_k < f_j$, for all $0 \le k < j \le r - k$,
(ii) $f_k \le f_{r-k+1}$, if $1 \le k < (r+1)/2$ and $h_{r-k+1} \ge 1$, with equality if and only if in addition $h_2 = 1$,
(iii) in case all e–element subsets of V belong to Δ, for $e < r$, then

$$f_k \ge \sum_{i=0}^{e} \binom{v-r+i-1}{i} \binom{r-i}{k-i}$$

for $k = 0, 1, \ldots, r$, and the following conditions are equivalent
(α) equality holds for some $k > e$
(β) equality holds for all k
(γ) $f_r = \binom{v-r+e}{e}$.

Proof. Part (i) follows directly from (7.5) and the nonnegativity (7.7) of h_k, together with elementary properties of binomial coefficients. For part (ii) a little more of the structure of h–vectors must be used, see Exercise 7.2.2.

Now, suppose as in (iii) that all e–element subsets of V are in Δ, that is, $f_k = \binom{v}{k}$ for $k = 0, 1, \ldots, e$. Using standard combinatorial identities such as $(-1)^y\binom{-x}{y} = \binom{x+y-1}{y}$ and the Vandermonde convolution formula $\binom{x+y}{k} = \sum_{i=0}^{k} \binom{x}{i} \binom{y}{k-i}$, (7.6) can then be developed as follows for $k \le e$

$$(7.8) \qquad h_k = \sum_{i=0}^{k} (-1)^{i+k} \binom{r-i}{k-i} \binom{v}{i} = \sum_{i=0}^{k} \binom{k-r-1}{k-i} \binom{v}{i} = \binom{v-r+k-1}{k}.$$

Inserting these values into (7.5) and using that $h_k \ge 0$ for $k > e$ we see that the inequality in (iii) arises from (7.5) by singling out the first $e+1$ terms. It is also evident by this argument that if

$$(\delta) \qquad h_{e+1} = h_{e+2} = \ldots = h_r = 0,$$

then equality in (iii) holds for all k. Thus, $(\delta) \Rightarrow (\beta) \Rightarrow (\gamma) \Rightarrow (\alpha)$, since ($\gamma$) means that equality holds for $k = r$. If equality holds in (iii) for some $k > e$ then (7.5) shows that

$h_{e+1} = h_{e+2} = \ldots = h_k = 0$. It is a consequence of the following lemma, in view of (7.7), that then also $h_{k+1} = h_{k+2} = \ldots = h_r = 0$, so (α) implies (δ). □

7.2.6 **Lemma.** Let $\mathcal{F} = \{\mathcal{R}(F_i) : 1 \le i \le t\}$, where $F_1, F_2, \ldots, F_t$ is a shelling of Δ. Then given $A \in \mathcal{F} - \{\emptyset\}$ there exists $x \in A$ such that $A - x \in \mathcal{F}$.

Proof. Suppose that $1 < g \le t$ and let $X_1, X_2, \ldots, X_s$ be the maximal proper subsets of $\mathcal{R}(F_g)$. They all belong to the subcomplex Δ_{g-1} since $\mathcal{R}(F_g)$ is the unique minimal face in $\Delta_g - \Delta_{g-1}$. Hence by Proposition 7.2.2 there are unique facets $F_{i_1}, F_{i_2}, \ldots, F_{i_s}$ satisfying $\mathcal{R}(F_{i_j}) \subseteq X_j \subseteq F_{i_j}$ and $i_j < g$ for $j = 1, 2, \ldots, s$. We have that $F_{i_j} \ne F_{i_k}$ when $j \ne k$, since if $X_j \subseteq F_{i_j}$ and $X_k \subseteq F_{i_j}$ then $\mathcal{R}(F_g) = X_j \cup X_k \subseteq F_{i_j}$ which contradicts 7.2.2. Assume that indices have been chosen so that $i_1 < i_2 < \ldots < i_s$. Then every proper subset of X_s belongs to $\Delta_{i_s - 1}$, so $\mathcal{R}(F_{i_s})$ cannot be strictly contained in X_s. Hence, $\mathcal{R}(F_{i_s}) = X_s \subset \mathcal{R}(F_g)$ and $|\mathcal{R}(F_g)| = |\mathcal{R}(F_{i_s})| + 1$. □

7.3. Matroid complexes

If $M = M(S)$ is a matroid of rank r on the finite set S, the family of all independent sets in M forms an $(r-1)$–dimensional simplicial complex, which we denote by $IN(M)$. Complexes of this kind are called *matroid complexes.* It is one of the first facts of matroid theory that a matroid complex is pure, and we will soon see that it is also shellable. In fact, matroid complexes can be characterized both in terms of purity and in terms of shellability (cf. Exercise 7.3.1 and Theorem 7.3.4).

A number of remarkable properties of matroids are revealed by, but not dependent on, assigning a linear order to the underlying point–set. Our approach to these results will be aided by the following definitions and conventions. By an *ordered matroid* $M(S,\omega)$ we will mean a matroid $M(S)$ together with a linear ordering ω of the underlying point set S. Let us agree to write a k–subset $A = \{x_1, x_2, \ldots, x_k\} \subseteq S$ as an ordered k–tuple $[x_1, x_2, \ldots, x_k]$ if and only if $x_1 < x_2 < \ldots < x_k$ under the order ω. The k–subsets of S are linearly ordered by the *lexicographic order* defined as follows: $[x_1, x_2, \ldots, x_k]$ precedes $[y_1, y_2, \ldots, y_k]$ if and only if $x_i = y_i$ for $i < e$ and $x_e < y_e$ for some position e. When in the sequel we compare bases of an ordered matroid it is always with respect to this induced linear order.

Recall that if B is a basis of $M(S)$ and $p \in S - B$, then there is a unique circuit $ci(B,p)$ contained in $B \cup p$. Dually, if $b \in B$ there is a unique bond $bo(B,b)$ contained in $(S-B) \cup b$. The *basic circuit* and *basic bond* are characterized as follows.

7.3.1. **Lemma.** $b \in ci(B,p) \iff (B - b) \cup p$ is a basis $\iff p \in bo(B,b)$.

Let C be a circuit of an ordered matroid $M = M(S,\omega)$ and c the least element in C. Then the set $C - c$ is called a *broken circuit.* A basis of M which contains no broken circuit will be referred to as an *nbc–basis.* This concept will figure prominently in later sections. For now it will be used for future reference in the following technical lemma, which provides the key to the results of this and the following section.

7.3.2. **Lemma.** Let $M(S,\omega)$ be an ordered matroid. Assume that the basis B precedes the basis C in the induced lexicographic order. Then $B \cap C \subsetneq A \cap C$ for some basis A which also precedes C, and such that $|A \cap C| = |C| - 1$. Further, if C is an *nbc*–basis then A can be chosen to be an *nbc*–basis.

Proof. Let $B = [b_1, b_2, \ldots, b_r], C = [c_1, c_2, \ldots, c_r]$ and assume that $b_i = c_i$ for $i = 1, 2, \ldots, e-1$, and $b_e \ne c_e$. Then $b_e < c_i$ for $i = e, e+1, \ldots, r$. By the basis exchange property there is an element $y \in C - B$ such that $A_1 = (C - y) \cup b_e$ is a basis. For the first claim we can let $A = A_1$ be this basis.

For the second claim, assume that C is an *nbc*–basis. If A_1 is not an *nbc*–basis then there is an element $a_1 \in S - A_1$ such that a_1 is the least element of the basic circuit $ci(A_1, a_1)$. Furthermore, b_e must belong to $ci(A_1, a_1)$, since otherwise C would contain the broken circuit $ci(A_1, a_1) - a_1$. Hence, $A_2 = (A_1 - b_e) \cup a_1$ is a basis, A_2 precedes A_1, $B \cap C \subseteq A_2 \cap C$ and $|A_2 \cap C| = r - 1$. If A_2 is not an *nbc*–basis the argument can be repeated until after a finite number of steps we reach a basis $A = A_k$ with the required properties. ◻

Disregarding the last sentence (about *nbc*–bases), Lemma 7.3.2 is equivalent to the following (cf. (7.3)):

7.3.3. **Theorem.** Let $M(S, \omega)$ be an ordered matroid. Then the ω–lexicographic order of bases of M is a shelling of $IN(M)$. In particular, all matroid complexes are shellable.

Thus every linear ordering of the ground set induces a shelling, and this rich supply of shellings in fact characterizes matroids.

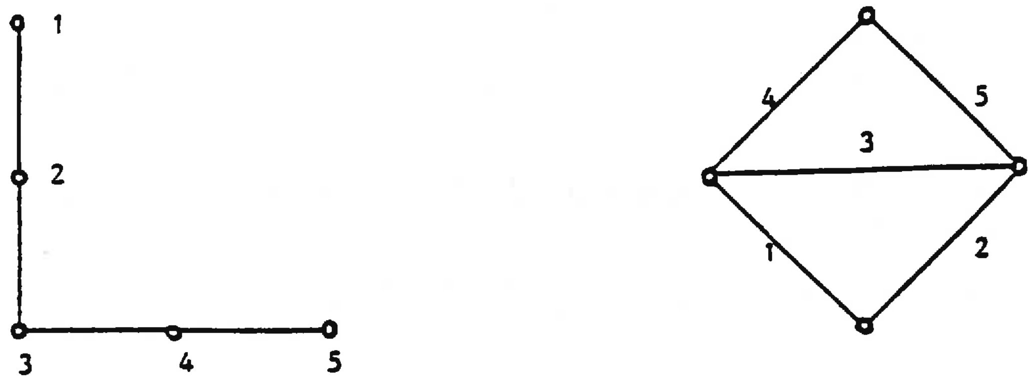
7.3.4. **Theorem.** A simplicial complex Δ is a matroid complex if and only if Δ is pure and every ordering of the vertices induces a shelling.

Proof. In one direction this is Theorem 7.3.3.

For the converse, suppose that a pure complex Δ is not a matroid complex. If V is the vertex set of Δ, then for some subset $U \subset V$ the induced subcomplex $\Delta_U = \{F \in \Delta : F \subseteq U\}$ is not pure (cf. Exercise 7.3.1). Let F be a facet of Δ_U of minimal dimension, and among the facets of Δ_U of dimension greater than $\dim F$ choose G such that $|F \cap G|$ is as large as possible. Now, order the vertex set V in such a way that the elements of $F - G$ come first, then the elements of G and the remaining elements last. Let $\tilde{G}$ be the first facet of Δ which contains G, and let $\tilde{F}$ be an arbitrarily chosen facet of Δ which contains F. Clearly, $\tilde{F} \cap U = F$ and $\tilde{G} \cap U = G$, since F and G are maximal in Δ_U. The chosen ordering of the vertices ensures that a facet H of Δ precedes $\tilde{G}$ if and only if $H \cap (F - G) \neq \emptyset$. In particular, $\tilde{F}$ precedes $\tilde{G}$. F and G are facets of Δ_U and $|F| < |G|$, so $|F \cap G| \leq |G| - 2$. Hence, $|\tilde{F} \cap \tilde{G}| \leq |\tilde{G}| - 2$. Assume that $\tilde{F} \cap \tilde{G} \subseteq H \cap \tilde{G} = \tilde{G} - g$ for some facet H of Δ which precedes $\tilde{G}$ and $g \in \tilde{G}$. Then $g \in \tilde{G} - \tilde{F}$, and $H - \tilde{G} = h \in F - G$. If $g \in \tilde{G} - G$, then $G \cup h \subseteq H \cap U$, hence $G \cup h \in \Delta_U$ which contradicts the maximality of G in Δ_U. If $g \in G - F$, then $G' = (G - g) \cup h$ is a face of Δ_U satisfying $\dim G' = \dim G > \dim F$ and $|F \cap G'| > |F \cap G|$, which contradicts the choice of G. Hence, such a facet H cannot exist, and the induced order is not a shelling. ◻

Having established the shellability of matroid complexes we will now find their shelling polynomials. Let B be a basis of an ordered matroid $M = M(S, \omega)$. An element $p \in S - B$ is said to be *externally active* in B if p is the least element in the basic circuit $ci(B, p)$. Otherwise p is *externally passive* in B. Dually, an element $p \in B$ is said to be *internally active* in B if p is the least element in the basic bond $bo(B, p)$. Otherwise p is *internally passive* in B. Denote by $EA(B), EP(B), IA(B)$ and $IP(B)$ the sets of externally active, externally passive, internally active and internally passive elements in B, respectively. We will call the number $i(B) = |IA(B)|$ the *internal activity* of B, and $e(B) = |EA(B)|$ the *external activity* of B. It is obvious that p is internally active in B if and only if p is externally active in the basis $S - B$ of the orthogonal ordered matroid $M^* = M^*(S, \omega)$.

7.3.5. **Example.** Let M be the matroid defined either by points and lines in affine space as in Figure 2a, or by the graph in Figure 2b.



(a) (b)

Figure 2

We will list the 8 bases of M in lexicographic order, and indicate for each basis which elements are active in it.

Basis	**Internally active**	**Externally active**
$B_1 = [1,2,4]$	1,2,4	–
$B_2 = [1,2,5]$	1,2	–
$B_3 = [1,3,4]$	1,4	–
$B_4 = [1,3,5]$	1	–
$B_5 = [1,4,5]$	1	3
$B_6 = [2,3,4]$	4	1
$B_7 = [2,3,5]$	–	1
$B_8 = [2,4,5]$	–	1,3

A picture of the complex $IN(M)$ appears in Figure 3. It takes the form of the surface of a triangular bipyramid together with the two interior triangles 145 and 245. It is instructive to observe how the sequence of bases $B_1, \ldots, B_8$ gives a shelling of $IN(M)$.

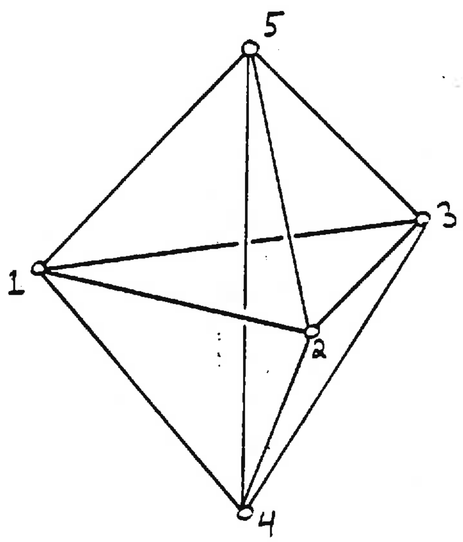


Figure 3.

Let $M = M(S,\omega)$ be an ordered matroid and consider again the lexicographic order of bases. It is in view of Lemma 7.3.1 clear that if B is a basis and $b \in B$ then $B - b$ is contained in a basis which precedes B if and only if b is internally passive in B. Thus,

$$\mathcal{R}(B) = IP(B). \tag{7.9}$$

The shelling polynomial $h_\Delta(x)$ of a matroid complex $\Delta = IN(M), M = M(S,\omega)$, is therefore equal to

$$h_\Delta(x) = \sum_B x^{|B-\mathcal{R}(B)|} = \sum_B x^{|IA(B)|} = \sum_B x^{i(B)}. \tag{7.10}$$

Dually, the shelling polynomial $h_{\Delta^*}(y)$ of the orthogonal matroid complex $\Delta^* = IN(M^*), M^* = M^*(S,\omega)$, is equal to

$$h_{\Delta^*}(y) = \sum_B y^{e(B)}.$$

In both cases the summation is over all bases B of M. It thus seems tempting to combine these two shelling polynomials associated with M into one polynomial of two variables

$$T_M(x,y) = \sum_B x^{i(B)} y^{e(B)}. \tag{7.11}$$

However, there is a complication. Whereas we know that the evaluations

$$T_M(x,1) = h_{IN(M)}(x) \text{ and } T_M(1,y) = h_{IN(M^*)}(y) \tag{7.12}$$

are independent of the order ω (Proposition 7.2.3), it is not immediately clear that $T_M(x,y)$ itself is independent of the ordering of S. We will soon see that this is the case, so that $T_M(x,y)$, called the *Tutte polynomial* of M, depends only on the matroid structure of M.

Before proving independence of the ordering, let us again look at the matroid M of Example 7.3.5. From the table of internally and externally active elements we conclude that its Tutte polynomial is

$$T_M(x,y) = x^3 + x^2 + x^2 + x + xy + xy + y + y^2 = x^3 + 2x^2 + y^2 + 2xy + x + y.$$

This implies that the shelling polynomial $h_\Delta(x)$ of the matroid complex $\Delta = IN(M)$ equals

$$h_\Delta(x) = T_M(x,1) = x^3 + 2x^2 + 3x + 2,$$

and for the orthogonal matroid $\Delta^* = IN(M^*)$

$$h_{\Delta^*}(y) = T_M(1,y) = y^2 + 3y + 4.$$

7.3.6. **Proposition.** Let $M = M(S,\omega)$ be an ordered matroid. Then the family of intervals $[IP(B), S - EP(B)]$, one for each basis B of M, partitions the Boolean algebra $\mathbf{2}^S$ of subsets of S.

Proof. We must prove for every subset $A \subseteq S$ that there is a basis B_A such that $IP(B_A) \subseteq A \subseteq S - EP(B_A)$, and that such a basis B_A is unique.

Let $A \subseteq S$. Then let X_A be the lexicographically greatest basis in the submatroid M_A which is induced on the subset A by M. The set X_A is independent in M and therefore there is a unique basis B_A such that $IP(B_A) \subseteq X_A \subseteq B_A$, by Proposition 7.2.2 and (7.9). Let $a \in A - B_A = A - X_A$, and assume that a is externally passive in B_A. This means that there is an element $b \in B_A$ such that $b < a$ and $(B_A - b) \cup a$ is a basis. In case $b \notin A$ this implies that $X_A \cup a$ is independent, which is impossible since X_A is a basis in M_A and $a \notin X_A$. If $b \in A$, then $b \in B_A \cap A = X_A$ and $(X_A - b) \cup a$ would be a basis in M_A strictly preceded by X_A, which contradicts the choice of X_A. Hence, all elements of $A - B_A$ are externally active in B_A. We have shown that $IP(B_A) \subseteq X_A \subseteq A \subseteq S - EP(B_A)$.

Next, suppose that $IP(B) \subseteq A \subseteq S - EP(B)$ for some basis B of M. Observe that

(7.13) $EA(B)$ is contained in the closure of $IP(B)$.

To see this, suppose that $p \in S - B$ is externally active in B and let C be the basic circuit ci (B,p) and C' the corresponding broken circuit ci $(B,p) - p$. If $q \in C'$ then $p < q$ and $(B - q) \cup p$ is a basis, so q is internally passive in B. Since p lies in the closure of C' and $C' \subseteq IP(B)$, statement (7.13) follows. This fact (7.13) implies that $X = B \cap A$ is a basis of M_A. Suppose that $X \neq X_A$, where X_A is defined as above. Let $X = \{x_1, x_2, \ldots, x_a\}, x_1 < x_2 < \ldots < x_a$, and $X_A = \{y_1, y_2, \ldots, y_a\}, y_1 < y_2 < \ldots < y_a$. Since X_A is the greatest basis of M_A there is an index e such that $x_i = y_i$ for $i = 1, 2, \ldots, e-1$ and $x_e < y_i$ for $i = e, e+1, \ldots, a$. By the basis exchange axiom $(X - x_e) \cup y_j$ is a basis for some $j \geq e$. But then y_j is externally passive in X, and hence also in B, contradicting the assumption $A \subseteq S - EP(B)$. Hence, $X = X_A$ and, since $IP(B) \subseteq X \subseteq B$, also $B = B_A$. □

7.3.7. **Theorem.** Let $M = M(S, \omega)$ be an ordered matroid with Tutte polynomial $T_M(x, y)$. Then

$$T_M(1+\xi, 1+\eta) = \sum_{A \subseteq S} \xi^{r(S)-r(A)} \eta^{r^*(S)-r^*(S-A)},$$

where r and r^* denote the rank functions of M and the orthogonal matroid M^*, respectively.

In particular, the Tutte polynomial does not depend on the ordering ω of S.

Proof. Let B be a basis of M and assume that $IP(B) \subseteq A \subseteq S - EP(B)$. The observation (7.13) that $EA(B)$ is contained in the closure of $IP(B)$ implies that $r(A) = |A \cap B|$, so $r(S) - r(A) = |B - A|$. Dually, $r^*(S) - r^*(S - A) = |(S - B) - (S - A)| = |A - B|$. Thus, using Proposition 7.3.6 we get

$$\sum_{A \subseteq S} \xi^{r(S)-r(A)} \eta^{r^*(S)-r^*(S-A)} =$$

$$= \sum_{B \text{basis}} \sum_{IP(B) \subseteq A \subseteq S - EP(B)} \xi^{|B-A|} \eta^{|A-B|} =$$

$$= \sum_{B \text{basis}} \sum_{j,k=0}^{\infty} \binom{i(B)}{j} \binom{e(B)}{k} \xi^j \eta^k =$$

$$= \sum_{B \text{basis}} (1+\xi)^{i(B)} (1+\eta)^{e(B)} = T_M(1+\xi, 1+\eta). \quad \square$$

The Tutte polynomial is treated in great detail in Chapter 6 of this volume, to which the reader is referred for more information.

Before ending this section, let us mention that (7.12) and Corollary 7.2.4 imply for the Euler characteristic of $IN(M)$ that $\chi(IN(M)) = (-1)^{r-1}T_M(0,1)$. Another expression will be given in Proposition 7.4.7.

7.4. Broken circuit complexes

Let $M = M(S,\omega)$ be an ordered matroid. Recall that when the least element of a circuit is deleted we call the remaining set a *broken circuit*. The family of all subsets of S that contain no broken circuit forms a simplicial complex which we denote $BC_\omega(M)$ and call the *broken circuit complex* of M. Note that $BC_\omega(M)$ is defined if and only if M is loopless, since if M has a loop then every subset of S contains the broken circuit $\emptyset$.

When discussing broken circuit complexes we may whenever convenient exclude the existence not only of loops but also of parallel elements in a matroid. Two elements $x, y \in S$ in a loopless matroid $M(S)$ are said to be *parallel* if $\text{rank}_M(\{x,y\}) = 1$. Parallelism is an equivalence relation. Define a rank function on the set $\bar{S}$ of equivalence classes by $\text{rank}_{\bar{M}}(\{X_1, X_2, \ldots, X_n\}) = \text{rank}_M(X_1 \cup X_2 \cup \ldots \cup X_n)$. This determines a matroid $\bar{M}(\bar{S})$ without loops or parallel elements, the *simplification* of M. If $M(S,\omega)$ is ordered let $\bar{M}(\bar{S},\bar{\omega})$ be ordered by the first elements in the respective parallelism classes. The following observation is straightforward.

7.4.1. **Proposition.** The two broken circuit complexes $BC_\omega(M)$ and $BC_{\bar{\omega}}(\bar{M})$ are isomorphic. □

The role which is played by the ordering ω in the construction of broken circuit complexes should perhaps be elucidated. Different orderings of the point set of a given matroid may yield nonisomorphic broken circuit complexes, as illustrated in Example 7.4.4 below. However, we will find that the important invariants of such complexes are independent of order.

Let $M(S,\omega)$ be an ordered loopless matroid with first element e. The family of all subsets of $S - e$ which contain no broken circuit will be called the *reduced broken circuit complex* and denoted $\overline{BC}_\omega(M)$. Let us gather some initial observations.

7.4.2. **Proposition.** Let $M = M(S,\omega)$ be an ordered loopless matroid of rank r. Then:
(*i*) $\overline{BC}_\omega(M) \subseteq BC_\omega(M) \subseteq IN(M)$,
(*ii*) $BC_\omega(M)$ is a pure $(r-1)$–dimensional complex whose facets are the *nbc*–bases of M,
(*iii*) $BC_\omega(M)$ is a cone over $\overline{BC}_\omega(M)$ with apex e,
(*iv*) $\overline{BC}_\omega(M)$ is a pure $(r-2)$– dimensional complex.

Proof. Suppose that $X \subseteq S$ contains no broken circuit. Then, a fortiori, X contains no circuit and hence $X \in IN(M)$. Being independent, X is included in some basis of M. Let B be the lexicographically first such basis. Suppose that B contains a broken circuit C. In this situation it would be possible to find elements y and z in S such that $y \notin B, C \cup y = \text{ ci }(B,y)$, $z \in C - X$ and $y < z$. Then $(B - z) \cup y$ would be a basis which contains X and precedes B , which contradicts the choice of B. Thus B cannot contain any broken circuit, that is, B is an *nbc*–basis. We have shown parts (*i*) and (*ii*).

If a basis B of M does not contain the first element e of S then B includes the broken circuit $\text{ ci }(B,e) - e$. Thus all *nbc*–bases contain e so that $BC_\omega(M)$ is a cone with apex e. Parts (*iii*) and (*iv*) now follow. □

If B is an *nbc*–basis, let us call $B - e$ a *reduced nbc–basis*. The following result is basic to this section.

7.4.3. **Theorem.** Let $M(S,\omega)$ be an ordered loopless matroid. Then $BC_\omega(M)$ and $\overline{BC}_\omega(M)$ are shellable. In both cases the ω-lexicographic ordering of facets (*nbc*-bases, resp., reduced *nbc*-bases) is a shelling.

Proof. For $BC_\omega(M)$ the result was proved in Lemma 7.3.2. By Proposition 7.4.2 (*iii*) and Exercise 7.2.1 the result follows also for $\overline{BC}_\omega(M)$. ▫

7.4.4. **Example.** Let $M = M(S)$ be the matroid on the set $S = \{1,2,3,4,5\}$ of Example 7.3.5. Under the natural ordering ω of S the broken circuits are $\{2,3\}, \{4,5\}$ and $\{2,4,5\}$ and the *nbc*-bases are $B_1 = \{1,2,4\}, B_2 = \{1,2,5\}, B_3 = \{1,3,4\}$ and $B_4 = \{1,3,5\}$. Under the ordering $\omega' : 1 < 2 < 4 < 3 < 5$ of S the broken circuits are $\{2,3\}, \{3,5\}$ and $\{2,4,5\}$ and the *nbc*-bases are $B_1 = \{1,2,4\}, B_2 = \{1,2,5\}, B_3 = \{1,3,4\}$ and $B_5 = \{1,4,5\}$. Thus, $BC_\omega(M)$ and $BC_{\omega'}(M)$ are nonisomorphic; the corresponding reduced complexes are illustrated in Figure 4. Observe here also how in both cases the respective lexicographic ordering of edges gives a shelling.

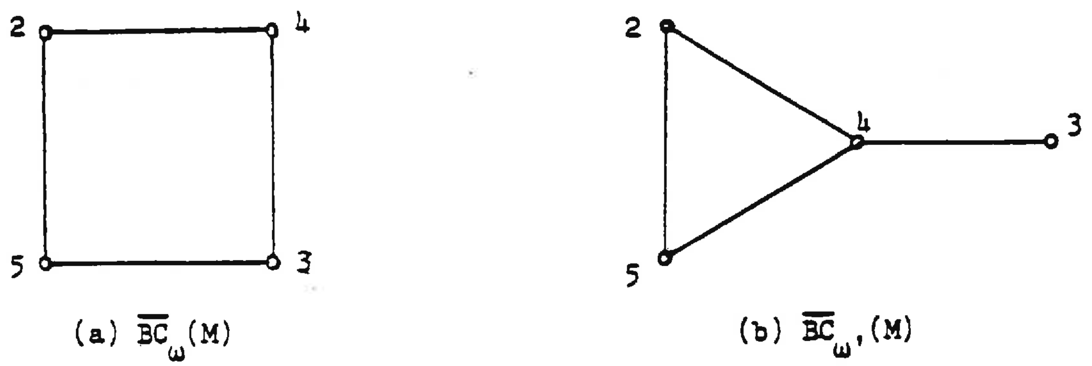


Figure 4.

We will now turn to the shelling polynomials of (reduced) broken circuit complexes and the related enumerative aspects. Suppose that B is an *nbc*-basis in an ordered loopless matroid $M(S,\omega)$ and $b \in B$. If $B - b$ is contained in a lexicographically smaller *nbc*-basis then, as in (7.9), $b \in IP(B)$. If on the other hand b is internally passive in B, then $B - b$ is contained in some basis A which precedes B and, as was shown in the proof of Proposition 7.4.2, the earliest basis which contains $B - b$ is an *nbc*-basis. Thus we have shown that in the ω-lexicographic shelling of $BC_\omega(M)$

$$\mathcal{R}(B) = IP(B). \tag{7.14}$$

Consequently, the shelling polynomial $h_\Delta(x)$ of $\Delta = BC_\omega(M)$ equals $\sum_B x^{i(B)}$ with summation over all *nbc*- bases B. But B contains no broken circuit if and only if $e(B) = 0$. Hence,

$$h_{BC_\omega(M)}(x) = T_M(x,0). \tag{7.15}$$

By similar reasoning it is straightforward to show directly, or it can be deduced from (7.15) via Proposition 7.4.2 (*iii*) and Exercise 7.2.1, that the shelling polynomial of the reduced broken circuit complex equals

$$h_{\overline{BC}_\omega(M)}(x) = \frac{1}{x}\, T_M(x,0). \tag{7.16}$$

Formula (7.15) implies that the face numbers of broken circuit complexes are independent of the ordering ω. See Example 7.4.4 for an illustration of this fact. For both orderings considered there the (non-isomorphic) broken circuit complexes have f-vector $(1, 5, 8, 4)$.

The face numbers of broken circuit complexes and their face enumerator $T_M(1+\lambda, 0)$ are among the most interesting numerical invariants in matroid theory. In the following we shall assume familiarity with the *Möbius function* $\mu(x, y)$ and the *characteristic polynomial* $p(L; \lambda) = \sum_{x \in L} \mu(\hat{0}, x)\lambda^{r-r(x)} = \sum_{k=0}^{r} w_k \lambda^{r-k}$ of a rank r geometric lattice L. For this, see e.g. Chapters 7 and 8 in White (1987) or Stanley (1986). The nonnegative integers $\bar{w}_k = (-1)^k w_k$ are called the *(unsigned) Whitney numbers of the first kind.*

7.4.5. **Proposition.** Let $M(S, \omega)$ be an ordered loopless matroid and L the corresponding geometric lattice. For $x \in L$, put $\mathbf{nbc}(x) = \{A \in BC_\omega(M) : \bar{A} = x\}$. Then

$$|\mathbf{nbc}(x)| = (-1)^{r(x)} \mu(\hat{0}, x).$$

Proof. We will show that $(-1)^{r(y)}|\mathbf{nbc}(y)|$ satisfies the same recursion as $\mu(\hat{0}, y)$. Since only the empty set spans the empty flat we have that $|\mathbf{nbc}(\hat{0})| = 1$, as required.

Assume that $x \neq \hat{0}$, let S' be the set points on the flat x and let $M'(S', \omega')$ be the restriction of the ordered matroid $M(S, \omega)$ to S' . It is straightforward to check that a subset of S' is a broken circuit in M' if and only if it is a broken circuit in M. Hence, $BC_{\omega'}(M') = \cup_{y \leq x} \mathbf{nbc}(y)$, and the union is disjoint. Since all members of $\mathbf{nbc}(y)$ have cardinality $r(y)$, being independent and spanning y, and since $BC_{\omega'}(M')$ is a cone, it follows from (7.2) that

$$\sum_{\hat{0} \leq y \leq x} (-1)^{r(y)} |\mathbf{nbc}(y)| = -\chi(BC_{\omega'}(M')) = 0.$$

The proof is complete. □

Summing the left-hand side in Proposition 7.4.5 over all flats x of rank k gives the total number of broken-circuit-free subsets of size k. The same summation for the right-hand side gives the Whitney number $\bar{w}_k$. Consequently we have proved:

7.4.6. **Theorem.** For a loopless matroid, the Whitney numbers of the first kind coincide with the face numbers of the broken circuit complex (induced by any order).

In view of (7.15) and Proposition 7.2.3 this result can be expressed:

(7.17) $$p(M; \lambda) = (-1)^r h_{BC_\omega(M)}(1-\lambda) = (-1)^r T_M(1-\lambda, 0).$$

To illustrate this result, consider once again the matroid M of Example 7.3.5. Its Tutte polynomial $T_M(x, y) = x^3 + 2x^2 + y^2 + 2xy + x + y$ was computed following (7.12), and we get $T_M(1-\lambda, 0) = 4 - 8\lambda + 5\lambda^2 - \lambda^3 = -p(M; \lambda)$. This should be compared to the f-vector (1, 5, 8, 4) of M's broken circuit complexes, see Example 7.4.4.

We end this section with the determination of the Euler characteristics of matroid and (reduced) broken circuit complexes. For this we will need the *beta invariant*

(7.18) $$\beta(M) = (-1)^{r(S)} \sum_{A \subseteq S} (-1)^{|A|} r(A),$$

of a matroid $M = M(S)$, discussed in Chapter 7 of White (1987). Also, we define the *Möbius invariant* $\bar{\mu}(M)$ by

(7.19) $$\bar{\mu}(M) = \begin{cases} |\mu_L(\hat{0}, \hat{1})|, & \text{if } M \text{ is loopless,} \\ 0, & \text{if } M \text{ has loops,} \end{cases}$$

where in the first case L is the lattice of flats of M. Clearly, $\tilde{\mu}(M) = \tilde{w}_r(L)$.

7.4.7. **Proposition.** Let M be a rank r matroid, which in parts (*ii*) and (*iii*) is supposed to be loopless and ordered by ω. Then,
(*i*) $\chi(IN(M)) = (-1)^{r-1}\tilde{\mu}(M^*)$,
(*ii*) $\chi(BC_\omega(M)) = 0$
(*iii*) $\chi(\overline{BC}_\omega(M)) = (-1)^r\beta(M)$.

Proof. Part (*ii*) is immediately clear, since $BC_\omega(M)$ is a cone. For the other parts we use the relation $\chi(\Delta) = (-1)^{r-1}h_\Delta(0)$ from Corollary 7.2.4.

For part (*i*), relations (7.12) and (7.17) give

$$h_{IN(M)}(0) = T_M(0,1) = T_{M^*}(1,0) = (-1)^{n-r}p(M^*;0) = \tilde{\mu}(M^*),$$

if M^* is loopless. If M^* has a loop then $IN(M)$ is a cone and both sides equal zero.

For part (*iii*), relation (7.16) and Theorem 7.3.7 give

$h_{\overline{BC}_\omega(M)}(0) = \frac{\partial T_M}{\partial x}(0,0)$

$= \sum_{A\subseteq S}\ (r(S) - r(A))(-1)^{r(S)-r(A)-1+r^*(S)-r^*(S-A)}$

$= (-1)^{r(S)}\sum_{A\subseteq S}\ (r(A) - r(S))(-1)^{|A|} = \beta(M).$ □

It is a consequence of the preceding that $\beta(M) \geq 0$ for all matroids M. For future reference we state the following properties of the beta invariant, see Chapter 7 of White (1987) for proofs.

7.4.8. **Proposition.** For any matroid $M = M(S)$ with $|S| \geq 2$:
(*i*) $\beta(M) = \beta(M^*)$, and
(*ii*) $\beta(M) > 0$ if and only if M is connected.

7.5. Application to matroid inequalities

The material developed in the preceding sections provide a good framework for dealing with certain inequalities for independence numbers and Whitney numbers of the first kind. This section, which can be skipped without loss of continuity, is devoted to this application. Below the inequalities considered lies the deeper and largely unsolved problem of understanding the h–vectors of matroid and broken circuit complexes.

Throughout this section r and n will always denote the rank and cardinality of the matroid M under consideration, and c_M (the *girth* of M) will be the smallest size of a circuit, that is, $c_M = \min\{|C| : C \text{ is a circuit in } M\}$.

We begin with the face numbers of matroid complexes, often called *independence numbers*. For a given matroid $M = M(S)$ let I_k denote the number of k–element independent subsets of $S, k = 0, 1, \ldots, r$. The number $b(M) = I_r$ of bases of M is of particular interest.

7.5.1. **Proposition.** Let M be a loopless matroid with $n > r$. Then
(*i*) $I_k < I_j$, for all $0 \leq k < j \leq r - k$,
(*ii*) $I_k \leq I_{r-k+1}$, if $1 \leq k < (r+1)/2$ and M has fewer than k isthmuses.

7.5.2. **Proposition.** Let M be a loopless matroid and $c = c_M$. Then

$$I_k \geq \sum_{i=0}^{c-1} \binom{n-r+i-1}{i} \binom{r-i}{k-i}$$

for $k = 0, 1, \ldots, r$, and equality holds for some $k \geq c$, or equivalently for all k, if and only if M is isomorphic to the direct product of the free matroid B_{r-c+1} and the $(c-1)$–uniform matroid $U_{c-1,n-r+c-1}$.

Proof. The inequalities result from letting $\Delta = IN(M)$ in Proposition 7.2.5. For 7.5.1 see Exercise 7.5.2.

Suppose that M is such that equality holds for all k in 7.5.2. Then $b(M) = \binom{n-r+c-1}{c-1}$, and in Proposition 7.2.5 we proved that the h–vector of $IN(M)$ must equal $(1, n-r, \ldots, \binom{n-r+c-2}{c-1}, 0, 0, \ldots, 0)$. The form of the Tutte polynomial $T_M(x,1) = x^r + (n-r)x^{r-1} + \ldots + \binom{n-r+c-2}{c-1}x^{r-c+1}$ reveals that M has exactly $r-c+1$ isthmuses (cf. Exercise 7.5.1) which together form a free submatroid B_{r-c+1} . Thus, $M = A \oplus B_{r-c+1}$, where A is a matroid of rank $c-1$ on $n-r+c-1$ points. Since $b(A) = b(M) = \binom{n-r+c-1}{c-1}$ we are forced to conclude that $A = U_{c-1,n-r+c-1}$. ◻

The preceding result shows in the particular case $k = r$ that for any loopless matroid M

$$b(M) \geq \binom{n-r+c-1}{c-1}. \tag{7.20}$$

Let M be a loopless matroid and let $(h_0, h_1, \ldots, h_r)$ be the h–vector of $IN(M)$. We know from (7.12) that $T_M(x,1) = \sum_{i=0}^{r} h_i x^{r-i}$, and from (7.6) and Proposition 7.4.7 that $h_0 = 1, h_1 = n-r$ and $h_r = \tilde{\mu}(M^*)$. In case M is connected more can be said about the h–vector.

7.5.3. **Proposition.** If M has no isthmus then $h_i \geq n-r$ for $i = 2, 3, \ldots, r-1$, and if in addition M is connected or else no connected component of M is a circuit then also $h_r \geq n-r$.

Proof. Let us first assume that $M = M(S)$ is connected. The only connected matroid on 2 elements is the 2–point circuit for which $\mathbf{h} = (1,1)$, so the boundary value is in order for an induction argument on n. Let $e \in S$. Since M is connected we know that either the contraction M/e or the restriction $M-e$ is connected (cf. White, 1986, p. 181). Let $(h_0^c, h_1^c, \ldots, h_{r-1}^c)$ and $(h_0^r, h_1^r, \ldots, h_r^r)$ be the h–vectors of $IN(M/e)$ and $IN(M-e)$, respectively. The Tutte polynomial identity $T_M(x,y) = T_{M/e}(x,y) + T_{M-e}(x,y)$ evaluated at $y = 1$ shows that $h_i = h_{i-1}^c + h_i^r$ for $i = 1, 2, \ldots, r$.

Suppose first that M/e is connected. Since M/e is of rank $(r-1)$ on $(n-1)$ elements the induction assumption at once gives $h_i^c \geq n-r$ for $i = 1, 2, \ldots, r-1$. Next, suppose that $M-e$ is connected. Then $h_i^r \geq n-r-1$ for $i = 1, 2, \ldots, r$, since $M-e$ is of rank r on $(n-1)$ elements. The orthogonal $(M/e)^*$ of the contraction, being the restriction $M^* - e$ of the connected matroid M^*, cannot contain a loop. Hence, $\tilde{\mu}((M/e)^*) > 0$, that is, $h_{r-1}^c \geq 1$. But that forces $h_i^c \geq 1$ also for $i = 1, 2, \ldots, r-2$, as a consequence of (7.7) and Lemma 7.2.6.

Thus, in either case it follows that $h_i \geq n-r$ for $i = 2, 3, \ldots, r$, and the connected case is settled.

Assume now that it has been shown for loopless matroids without isthmuses having $p-1$ connected components that $h_i \geq n-r$ for $i = 1,2,\ldots,r-1$ and $h_r \geq 1$. Suppose that our matroid M har p components, one of which is M_1. Then $M = M_1 \oplus M_2$ where M_2 has $p-1$ components. If r_j, n_j and $(h_0^j, h_1^j, \ldots, h_{r_j}^j)$ denote the rank, cardinality and h–vector of $M_j, j = 1,2$, then we know that $r_1 + r_2 = r$, $n_1 + n_2 = n$, $h_i^1 \geq n_1 - r_1 \geq 1$ for $i = 1,2,\ldots,r_1$ and $h_i^2 \geq n_2 - r_2$ for $i = 1,2,\ldots,r_2-1, h_{r_2}^2 \geq 1$. A comparison of coefficients in the Tutte polynomial formula $T_M(x,1) = T_{M_1}(x,1) \cdot T_{M_2}(x,1)$ shows that $h_i = \sum_{i=j+k} h_j^1 \, h_k^2$. Thus we arrive at the desired conclusion: $h_i \geq n_1 - r_1 + n_2 - r_2 = n - r$ for $i = 1,2,\ldots,r-1$.

In case no connected component of M is a circuit then $n_1 - r_1 \geq 2$, and we can assume that $h_{r_2}^2 \geq n_2 - r_2 \geq 2$ in the induction assumption of the preceding paragraph. Thus, $h_r = h_{r_1}^1 \, h_{r_2}^2 \geq (n_1 - r_1) \cdot (n_2 - r_2) \geq n_1 - r_1 + n_2 - r_2 = n - r$. □

7.5.4. **Proposition.** Let M be a matroid without loops or isthmuses and $c = c_M$. Then

$$I_k \geq \sum_{i=0}^{c-1} \binom{n-r+i-1}{i} \binom{r-i}{k-i} + (n-r)\binom{r-c+1}{k-c}$$

for $k = 0,1,\ldots,r-1$.

If M is connected or else no connected component of M is a circuit then the formula holds also for $k = r$, that is,

$$b(M) \geq \binom{n-r+c-1}{c-1} + (n-r)(r-c+1).$$

Proof. We know that $h_i = \binom{n-r+i-1}{i}$ for $i = 0,1,\ldots,c-1$ by (7.8), and $h_i \geq n-r$ for $i = c, c+1, \ldots, r-1, (r)$ by 7.5.3, so the result follows from formula (7.5). □

Taking $c = 2$ in the above formulas we get

$$I_k \geq \binom{r}{k} + (n-r)\binom{r}{k-1}, k = 0,1,\ldots,r-1, \text{ and} \tag{7.21}$$

$$b(M) \geq 1 + r(n-r), \tag{7.22}$$

which hold for all matroids M without loops and isthmuses except that (7.22) fails by a trifle for a few products involving circuits (cf. Exercise 7.5.7). For *connected* matroids it can be shown that equality holds for all $k \leq r$ in (7.21), or equivalently in (7.22), if and only if M is isomorphic to the parallel connection of an $(r+1)$–point circuit and an $(n-r)$–point atom.

Taking $c = 3$ in Proposition 7.5.4 we get formulas which are valid for all connected simple matroids.

We will now turn our attention to the face numbers of broken circuit complexes. Recall from Theorem 7.4.6 that these are the Whitney numbers of the first kind $\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_r$. In particular, $\tilde{w}_r = \tilde{\mu}(M)$, the Möbius invariant (7.19). It will be assumed that all matroids are *simple*, i.e. lack loops and parallel elements. This results in no lack of generality (cf. Proposition 7.4.1).

7.5.5. **Proposition.** Let M be a simple matroid with $n > r$. Then
(*i*) $\tilde{w}_k < \tilde{w}_j$, for all $0 \le k < j \le r - k$,
(*ii*) $\tilde{w}_k \le \tilde{w}_{r-k+1}$, if $2 \le k < (r+1)/2$ and M has fewer than k connected components, or if $k = 1, 4 \le r \le n-2$ and M is connected.

7.5.6. **Proposition.** Let M be a simple matroid and $c = c_M$. Then

$$\tilde{w}_k \ge \sum_{i=0}^{c-2} \binom{n-r+i-1}{i}\binom{r-i}{k-i}$$

for $k = 0, 1, \ldots, r$, and equality holds for some $k \ge c-1$, or equivalently for all k, if and only if M is isomorphic to the direct product of the free matroid B_{r-c+1} and the $(c-1)$–uniform matroid $U_{c-1,n-r+c-1}$.

Taking $k = r$ in Proposition 7.5.6 we obtain

$$\tilde{\mu}(M) \ge \binom{n-r+c-2}{c-2}. \tag{7.23}$$

It also follows (since $c \ge 3$ for simple matroids) that

$$\tilde{w}_k \ge \binom{r}{k} + (n-r)\binom{r-1}{k-1} \tag{7.24}$$

for $k = 0, 1, \ldots, r$, with equality exclusively characterizing the direct products of free matroids and lines. It is interesting that the lower bounds 7.5.2 and 7.5.6 are attained by precisely the same class of matroids.

Proof. These results arise from applying Proposition 7.2.5 to a broken circuit complex of M. For 7.5.5 see Exercise 7.5.3.

We need here only verify the characterization of equivalence in 7.5.6. If $M = B \oplus U$ where $B = B_{r-c+1}$ and $U = U_{c-1,n-r+c-1}$ then $\tilde{w}_r = \tilde{\mu}(M) = \tilde{\mu}(B)\tilde{\mu}(U) = \tilde{\mu}(U) = \binom{n-r+c-2}{c-2}$, which according to 7.2.5 implies equivalence in 7.5.6 for all k.

Suppose now that M is such that equivalence holds in 7.5.6 for all k, in particular then $\tilde{\mu}(M) = \binom{n-r+c-2}{c-2}$. While proving Proposition 7.2.5 we showed that this implies that the Tutte polynomial $T_M(x,0)$, being the shelling polynomial of the broken circuit complex, equals $T_M(x,0) = x^r + (n-r)x^{r-1} + \ldots + \binom{n-r+c-3}{c-2}x^{r-c+2}$. From the form of $T_M(x,0)$ we can deduce that M is the direct product of $r-c+2$ connected simple matroids $M_1, M_2, \ldots, M_{r-c+2}$ (cf. Exercise 7.5.1). Assume that M_i is of rank r_i and cardinality n_i for $i = 1, 2, \ldots, r-c+2$. Clearly, $\tilde{\mu}(M_i) \le \binom{n_i-1}{r_i-1}$ since there are $\tilde{\mu}(M_i)$ *nbc*–bases in M_i all of which contain the first element under some ordering. From the same viewpoint it is apparent that $\tilde{\mu}(M_i) = \binom{n_i-1}{r_i-1}$ if and only if M_i is r_i–uniform. We get that

$$\tilde{\mu}(M) = \Pi_{i=1}^{r-c+2}\,\tilde{\mu}(M_i) \le \Pi_{i=1}^{r-c+2}\binom{n_i-1}{r_i-1} \le \binom{n-r+c-2}{c-2} = \tilde{\mu}(M).$$

But $\binom{a_1}{b_1}\binom{a_2}{b_2} \le \binom{a_1+a_2}{b_1+b_2}$ with equality if and only if $a_1 = b_1 = 0$ or $a_2 = b_2 = 0$. Thus $n_i = r_i = 1$ for all i except one, say $i = 1$, for which $n_1 = n - r + c - 1, r_1 = c - 1$ and $\tilde{\mu}(M_1) = \binom{n_1-1}{r_1-1}$. So $M_1 = U_{c-1,n-r+c-1}$ and $M_2 \oplus \ldots \oplus M_{r-c+2} = B_{r-c+1}$. □

Let $(h_0, h_1, \ldots, h_r)$ be the h–vector of a broken circuit complex of a simple matroid M. Thus $T_M(x,0) = h_0x^r + h_1x^{r-1} + \ldots + h_r$. We know from (7.6), Proposition 7.4.7 and Exercise 7.2.1 that in general $h_0 = 1$, $h_1 = n - r$, $h_{r-1} = \beta(M)$ and $h_r = 0$. However, as was the case for the matroid complex, when M is connected more can be said.

7.5.7. **Proposition.** If M is connected then $h_i \geq n - r$ for $i = 2,3,\ldots,r-2$, and $h_{r-1} \geq 1$.

Proof. $\beta(M) \geq 1$ is equivalent with M being connected (Proposition 7.4.8). For the other inequalities see Brylawski (1977b), Theorem 3.1.2. □

7.5.8. **Proposition.** Let M be a connected simple matroid and $c = c_M$. Then

$$\tilde{w}_k \geq \sum_{i=0}^{c-2} \binom{n-r+i-1}{i}\binom{r-i}{k-i} + (n-r)\binom{r-c+2}{k-c+1} \text{ for } k = 0,1,\ldots,r-2,$$

$$\tilde{w}_{r-1} \geq \sum_{i=0}^{c-2} (r-i)\binom{n-r+i-1}{i} + (n-r)[\binom{r-c+2}{2} - 1] + \beta(M) \text{ , and}$$

$$\tilde{\mu}(M) \geq \binom{n-r+c-2}{c-2} + (n-r)(r-c) + \beta(M).$$

Proof. The inequalities arise from (7.5), that is,

$$\tilde{w}_k = \sum_{i=0}^{r} h_i\binom{r-i}{k-i}, k = 0,1,\ldots,r;$$

since $h_i = \binom{n-r+i-1}{i}$ for $i = 0,1,\ldots,c-2$ by (7.8), and $h_i \geq n-r$ for $i = c-1,c,\ldots,r-2$, by 7.5.7. (The last two inequalities, the $k = r-1$ and $k = r$ cases, actually require that $c \leq r$. If $c > r$ then M is uniform and $\tilde{w}_{r-1} = \binom{n}{r-1}, \tilde{\mu}(M) = \binom{n-1}{r-1}$.) □

By taking $c = 3$ in the above formulas we get inequalities which hold for all connected simple matroids, namely

$$\tilde{w}_k \geq \tbinom{r}{k} + (n-r)\tbinom{r}{k-1}, \text{ for } k = 0,1,\ldots,r-2,$$

$$\text{(7.25)} \quad \tilde{w}_{r-1} \geq r + (n-r)[\tbinom{r}{2} - 1] + \beta(M), \text{ and}$$

$$\tilde{\mu}(M) \geq 1 + (n-r)(r-2) + \beta(M).$$

To eliminate the appearance of $\beta(M)$, the last two of these formulas can be replaced by

$$\text{(7.26)} \quad \tilde{w}_{r-1} \geq r + (n-r)\tbinom{r}{2} - 1, \text{ and}$$

$$\tilde{\mu}(M) \geq (n-r)(r-1),$$

which hold for all connected simple matroids with the exception of the parallel connection of three 3–point lines for which $r = 4$, $n = 7$, $\tilde{\mu} = 8$ and $\tilde{w}_3 = 20$, see Brylawski (1977b). The connected simple matroids which achieve equality in (7.25) and (7.26) have not been characterized; some examples are parallel connections of circuits and lines, the Fano projective plane and the complete graph K_4.

7.6. Order complexes of geometric lattices

Let L be a finite geometric lattice with proper part $\bar{L} = L - \{\hat{0}, \hat{1}\}$. The chains $x_0 < x_1 < \ldots < x_k$ of $\bar{L}$ are the simplices of a simplicial complex $\Delta(\bar{L})$, called the *order complex* of L. If rank $(L) = r$, then $\Delta(\bar{L})$ is a pure $(r-2)$-dimensional complex.

We assume familiarity with the cryptomorphic correspondence between simple matroids $M = M(S)$ and geometric lattices L, see Section 3.4 of White (1986). Under this correspondence the ground set S is identified with the set L^1 of atoms of L. For $0 \leq k \leq r =$ rank (L) let $L^k = \{x \in L : r(x) = k\}$, and for $x \in L^k$ let ϖx denote the corresponding flat $\varpi x = \{p \in L^1 : p \leq x\}$ in M. To ordered matroids $M = M(S, \omega)$ correspond *atom-ordered geometric lattices* (L, ω), i.e., ω is a linear ordering of L^1.

Let (L, ω) be an atom-ordered geometric lattice. Denote by $Cov(L) \subseteq L \times L$ the set of *coverings*, i.e., pairs (x, y) such that if $x \in L^i$ then $y > x$ and $y \in L^{i+1}$. An *edge-labeling* $\lambda = \lambda_\omega : Cov(L) \rightarrow L^1$ is defined by the rule

$$\lambda(x, y) = \min_\omega(\varpi y - \varpi x). \tag{7.27}$$

Note that the label $\lambda(x, y)$ is well-defined since the indicated set of atoms must be nonempty. Also, the labeling $\lambda = \lambda_\omega$ of course depends on ω.

7.6.1 **Example.** Figure 5(a) shows the geometric lattice of flats of the matroid from Example 7.3.5. Figure 5(b) shows the edge-labeling induced by the natural ordering $1 < 2 < 3 < 4 < 5$ of its atoms.

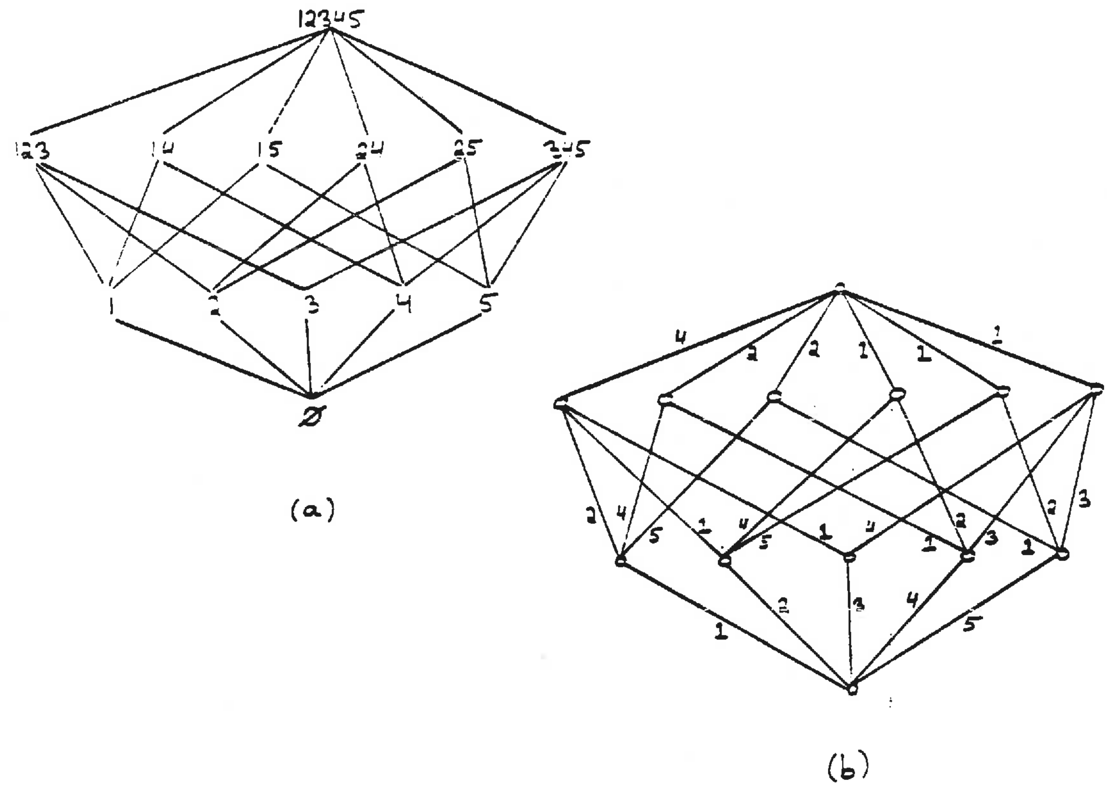


Figure 5.

The edge-labeling (7.27) induces a labeling of unrefinable chains. If the chain $\mathbf{c} : x_0 < x_1 < \ldots < x_k$ is unrefinable, meaning that (x_{i-1}, x_i) is a covering for $1 \leq i \leq k$, let

(7.28) $$\lambda(\mathbf{c}) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k)).$$

This label $\lambda(\mathbf{c})$ is an ordered k-tuple of atoms. As an (unordered) k-subset of atoms, $\lambda(\mathbf{c})$ is independent (cf. Exercise 7.6.3). Let us call $\lambda(\mathbf{c})$ *increasing* if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \ldots < \lambda(x_{k-1}, x_k)$, where the comparisons are made with respect to the order ω of L^1. If all inequalities go the other way $\lambda(\mathbf{c})$ is *decreasing*.

The crucial combinatorial properties of this labeling λ will now be stated.

7.6.2. **Lemma.** Let $x, y \in L$ with $x < y$, and let $\mathcal{M}_{x,y} = \{$ unrefinable chains $x = x_0 < x_1 < \ldots < x_k = y\}$. Then
(*i*) there exists a unique chain $\mathbf{c}_{x,y} \in \mathcal{M}_{x,y}$ with increasing label,
(*ii*) $\lambda(\mathbf{c}_{x,y})$ is lexicographically least in the set $\{\lambda(\mathbf{c}) : \mathbf{c} \in \mathcal{M}_{x,y}\}$.

Proof. Put $x_0 = x$, and define recursively elements $p_i \in L^1$ and $x_i \in L^{r(x_0)+i}$ by $p_i = \min_\omega(\Box y - \Box x_{i-1})$ and $x_i = x_{i-1} \vee p_i$. This recursive definition will end with $x_k = y$, where $k = r(y) - r(x)$. Let $\mathbf{c}_{x,y} : x_0 < x_1 < \ldots < x_k$. Then $\lambda(\mathbf{c}_{x,y}) = (p_1, p_2, \ldots, p_k)$, which by construction is increasing.

Suppose that $\mathbf{c} : x = y_0 < y_1 < \ldots < y_k = y$, with $y_i = x_i$ for $0 \leq i \leq e-1$ and $y_e \neq x_e$. Then $\lambda(\mathbf{c}) = \{p_1, \ldots, p_{e-1}, q_e, \ldots, q_k\}$ with $q_e \neq p_e$. The construction shows that in fact $p_e < q_e$, and that $p_e = q_f$ for some $f > e$. Hence $\lambda(\mathbf{c})$ is lexicographically greater than $\lambda(\mathbf{c}_{x,y})$, and $\lambda(\mathbf{c})$ is not increasing. □

The facets of the order complex $\Delta(\bar{L})$ are the maximal chains in $\bar{L}$. Extending these by $\hat{0}$ and $\hat{1}$ we get an identification with the set $\mathcal{M} = \mathcal{M}_{\hat{0},\hat{1}}$ of maximal chains in L. This leads to the main result of this section.

7.6.3. **Theorem.** Let (L, ω) be an atom-ordered geometric lattice. Then the ω-lexicographic order of labels $\lambda(\mathbf{c})$ determines a shelling order of the set $\mathcal{M}$ of maximal chains (facets of $\Delta(\bar{L})$). In particular, $\Delta(\bar{L})$ is shellable.

Proof. By (7.3) the following must be verified: If $\mathbf{c}, \mathbf{d} \in \mathcal{M}$ with $\lambda(\mathbf{d}) < \lambda(\mathbf{c})$, then there exists $\mathbf{e} \in \mathcal{M}$ and $x \in \mathbf{c}$ such that $\lambda(\mathbf{e}) < \lambda(\mathbf{c})$ and $\mathbf{c} \cap \mathbf{d} \subseteq \mathbf{c} \cap \mathbf{e} = \mathbf{c} - x$.

Suppose that $\mathbf{c} : \hat{0} = x_0 < x_1 < \ldots < x_r = \hat{1}$, and $\mathbf{d} : \hat{0} = y_0 < y_1 < \ldots < y_r = \hat{1}$, and $\lambda(\mathbf{d}) < \lambda(\mathbf{c})$. Suppose furthermore that $x_i = y_i$ for $0 \leq i \leq f$, and that $x_{f+1} \neq y_{f+1}$. Let g be the least integer greater than f such that $x_g = y_g$ (g is well-defined since $x_r = y_r$). Then $g - f \geq 2$ and $f < i < g$ implies that $x_i \neq y_i$. See Figure 6.

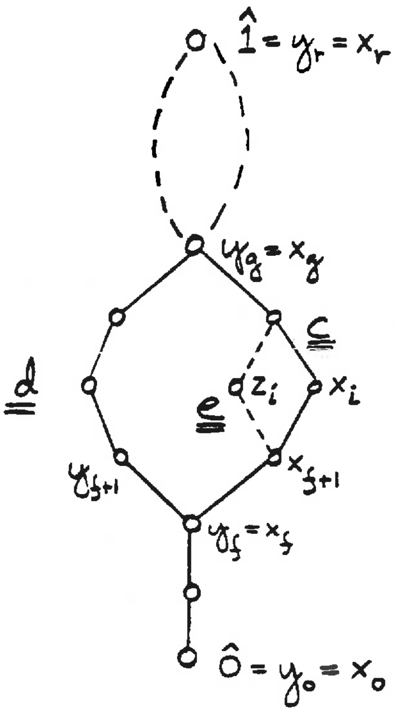


Figure 6.

The fact that $\lambda(\mathbf{d}) < \lambda(\mathbf{c})$ shows that $x_f < x_{f+1} < \ldots < x_g$ cannot have the lexicographically least label in the set $\mathcal{M}_{x_f,x_g}$. Hence by Lemma 7.6.2 there exists $f < i < g$ such that $\lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})$. Again by the lemma there exists a unique $z_i \in L^i$ such that $x_{i-1} < z_i < x_{i+1}$ and $\lambda(x_{i-1},z_i) < \lambda(z_i,x_{i+1})$. Replace x_i in c by z_i. This gives a new maximal chain e such that $\mathbf{c} \cap \mathbf{d} \subseteq \mathbf{c} \cap \mathbf{e} = \mathbf{c} - x_i$, and (by the lemma) $\lambda(\mathbf{e}) < \lambda(\mathbf{c})$. □

Let $\mathbf{c} : x_1 < x_2 < \ldots < x_{r-1}$ be a maximal chain in $\bar{L}$. As before, c may tacitly be identified with its extension by $x_0 = \hat{0}$ and $x_r = \hat{1}$ to a maximal chain in L. The shellability proof (and in essence Lemma 7.6.2) shows that there exists a maximal chain e such that $\lambda(\mathbf{e}) < \lambda(\mathbf{c})$ and $\mathbf{e} \cap \mathbf{c} = \mathbf{c} - x_i$ if and only if $\lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})$. Hence, the restriction operator induced by the shelling in Theorem 7.6.3 is given by

(7.29)
$$\mathcal{R}(\mathbf{c}) = \{x_i \in \mathbf{c} : \lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})\}.$$

Let us write simply $\mu(L)$ for the Möbius function value $\mu(\hat{0},\hat{1})$, computed over the geometric lattice L. A well-known theorem of P. Hall gives that $\mu(L)$ is equal to the number of odd cardinality chains in $\bar{L}$ minus the number of even cardinality chains in $\bar{L}$ (including the empty chain), see Rota (1964) or Stanley (1986). Part (*i*) of the following proposition is a restatement of Hall's theorem for L. Part (*ii*) is then implied via (7.29) and Corollary 7.2.4.

7.6.4. **Proposition. Let L be a geometric lattice of rank r. Then**
(*i*) $\chi(\Delta(\bar{L})) = \mu(L)$,
(*ii*) $\mu(L) = (-1)^r \cdot \#\{\mathbf{c} \in \mathcal{M} : \lambda(\mathbf{c})$ is decreasing$\}$, for any labeling $\lambda = \lambda_\omega$ induced by an atom-ordering ω as in (7.27) and (7.28).

For an illustration of this result, take a look at the rank 3 geometric lattice in Figure 5. Direct computation of the Möbius function, using its recursive definition, gives that $\mu(L) = -4$. The 4 decreasing labels of maximal chains are (4,2,1), (4,3,1), (5,2,1) and (5,3,1).

The combinatorics of the edge-labelings (7.27) of geometric lattices L extends to so called *rank-selected subposets*, obtained by deleting an arbitrary set of rank levels L^k

from L. The results developed so far in this section have more general versions for rank-selected subposets and their order complexes, see Exercise 7.6.5.

Combining Proposition 7.4.5 and part (*ii*) of Proposition 7.6.4 we find that the number of *nbc*–bases in L equals the number of maximal chains with decreasing label. This fact can also be established by an explicit bijection.

For an atom-ordered geometric lattice (L,ω) of rank r, let $\mathcal{DM}$ denote the set of maximal chains in L with decreasing labels, and let $\mathbf{nbc} = \mathbf{nbc}(\hat{1})$ denote the set of *nbc*–bases of L. Suppose that $\mathbf{c} \in \mathcal{DM}$, with $\lambda(\mathbf{c}) = (\lambda_1, \lambda_2, \ldots, \lambda_r), \lambda_1 > \lambda_2 > \ldots > \lambda_r$. Then, clearly,

$$\varphi(\mathbf{c}) = \{\lambda_r, \lambda_{r-1}, \ldots, \lambda_1\} \text{ is an } nbc\text{-basis}. \tag{7.30}$$

Conversely, suppose that $B = \{b_1, b_2, \ldots, b_r\} \in \mathbf{nbc}, b_1 < b_2 < \ldots < b_r$, and construct the maximal chain $\psi(B) : \hat{0} < b_r < (b_r \vee b_{r-1}) < (b_r \vee b_{r-1} \vee b_{r-2}) < \ldots < (b_r \vee \ldots \vee b_1) = \hat{1}$. Then

$$\lambda(\psi(B)) = (b_r, b_{r-1}, \ldots, b_1), \text{ hence } \psi(B) \in \mathcal{DM}. \tag{7.31}$$

Since $\varphi \circ \psi(B) = B$ for all $B \in \mathbf{nbc}$, and conversely, we conclude that φ and ψ are bijections $\mathcal{DM} \leftrightarrow \mathbf{nbc}$.

Statement (7.31) is a consequence of

$$\text{if } A \in BC_\omega(L), \text{then } \min_\omega \bar{A} \in A, \tag{7.32}$$

i.e., the least element in A is also the least element in the flat spanned by A, which in turn follows directly from the definitions. Statement (7.32) also implies that if $B = \{b_1, b_2, \ldots, b_r\} \in \mathbf{nbc}, b_1 < b_2 < \ldots < b_r$, and if $\pi \in S_r$ is a permutation, then

$$\psi_\pi(B) \in \mathcal{DM} \text{ if and only if } \pi = \text{id}, \tag{7.33}$$

where $\psi_\pi(B) : \hat{0} < b_{\pi(r)} < (b_{\pi(r)} \vee b_{\pi(r-1)}) < \ldots < (b_{\pi(r)} \vee \ldots \vee b_{\pi(1)}) = \hat{1}$.

7.7. Homology of shellable complexes

This section will review the construction of simplicial homology and the basic facts about the homology of a shellable complex. The presentation is essentially self-contained and should be accessible to readers without a background in algebraic topology. Readers having such a background can proceed directly to Theorem 7.7.2.

Let Δ be a simplicial complex on vertex set V. We will temporarily assume that V is linearly ordered, but the particular order chosen is of no significance for the end product. Let us agree to permit a nonvoid face $F = \{v_0, v_1, \ldots, v_k\}$ of Δ to be written $F = [v_0, v_1, \ldots, v_k]$ if and only if $v_0 < v_1 < \ldots < v_k$ in the given order of V. Then let $C_k(\Delta)$ denote the free Abelian group generated by the symbols $[v_0, v_1, \ldots, v_k]$, i.e., by the set of k–dimensional faces of Δ written in canonical form. The elements of $C_k(\Delta)$ are formal linear combinations with integer coefficients of k–dimensional faces. Thus, $C_{-1}(\Delta) \cong \mathbf{Z}, C_0(\Delta) \cong \mathbf{Z}^{|V|}$, the direct sum of $|V|$ copies of $\mathbf{Z}$, and $C_k(\Delta) = 0$ for $k < -1$ and $k > d = \dim \Delta$. Let group homomorphisms

$$\partial_k : C_k(\Delta) \rightarrow C_{k-1}(\Delta)$$

be defined on the basis elements by

$$\partial_k[v_0, v_1, \ldots, v_k] = \sum_{i=0}^{k} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k], \tag{7.34}$$

and extend linearly to all of $C_k(\Delta)$, for $k = 0, 1, \ldots, d$. Put $\partial_k = 0$ otherwise. Here the hat symbol "$\wedge$" has the meaning that what stands underneath should be deleted. Thus, for instance $\partial_0[v_0] = \emptyset$ and $\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$.

The elements ρ of $C_k(\Delta)$ such that $\partial_k(\rho) = 0$ are called k–dimensional *cycles*, they form a subgroup denoted by $Z_k(\Delta)$. The elements σ of $C_k(\Delta)$ which satisfy $\sigma = \partial_{k+1}(\tau)$ for some $\tau \in C_{k+1}(\Delta)$ are called k–dimensional *boundaries* and form a subgroup $B_k(\Delta)$. It is easy to verify that $\partial_k \circ \partial_{k+1} = 0$ for all $k \in \mathbf{Z}$. Thus, $B_k(\Delta) \subseteq Z_k(\Delta)$ for all $k \in \mathbf{Z}$. The quotient group

$$H_k(\Delta) = Z_k(\Delta)/B_k(\Delta) \tag{7.35}$$

is the k–dimensional *homology group* of Δ with integer coefficients.

The homology groups we have defined are usually referred to as "reduced" homology and denoted "$\tilde{H}_k(\Delta)$". Since we have no occasion to consider any other variation of homology, we adhere to the simpler terminology. A technical feature of this definition is that $H_{-1}(\Delta) \cong \mathbf{Z}$ for the "empty complex" $\Delta = \{\emptyset\}$, while $H_{-1}(\Delta) = 0$ as soon as there are vertices (nondegenerate complexes). Also, the rank of $H_0(\Delta)$ is one less than the number of connected components of Δ.

The rank of the Abelian group $H_k(\Delta)$ is called the k-th *Betti number* of the complex Δ. By the *rank* of a finitely generated Abelian group we mean the maximum number of linearly independent elements of infinite order.

A complex Δ is called *acyclic* (*over* $\mathbf{Z}$) if $H_k(\Delta) = 0$ (i.e., $H_k(\Delta)$ is isomorphic to the trivial group) for all $k \in \mathbf{Z}$. The following two facts will be needed:

(7.36) a cone is acyclic, and

(7.37) if $\Delta_1, \Delta_2$and $\Delta_1 \cap \Delta_2$ are acyclic complexes then $\Delta_1 \cup \Delta_2$ is also acyclic.

Both facts are completely elementary given some knowledge of topology. For instance, (7.37) follows from the Mayer–Vietoris long exact sequence. They are also elementary in the sense that straightforward proofs directly from the definitions are easy.

7.7.1. **Lemma.** Given a shelling of a complex Δ, let $\Delta' = \Delta - \{$facets F such that $\mathcal{R}(F) = F\}$, where $\mathcal{R}(F)$ is the induced restriction operator. Then Δ' is acyclic.

Proof. The first observation to be made is that the ordering of facets of Δ, restricted to the facets in Δ', gives a shelling of Δ'. The second observation is that the restriction operator $\mathcal{R}(F)$ induced by this shelling of Δ' coincides with the original one, in particular then $\mathcal{R}(F) \neq F$ for all facets $F \in \Delta'$. These observations are simple consequences of the definition (7.3).

We will prove that Δ' is acyclic inductively using its shelling $F_1, F_2, \ldots, F_t$. Let $\bar{F}_i = \{G \in \Delta' : G \subseteq F_i\}$, and $\Delta_i = \bar{F}_1 \cup \bar{F}_2 \cup \ldots \cup \bar{F}_i, 1 \leq i \leq t$. We have that $\bar{F}_i$ is a cone with any of its vertices as apex, and $\Delta_{i-1} \cap \bar{F}_i$ is a cone with any vertex in $F_i - \mathcal{R}(F_i)$ as apex. Therefore by (7.36), $\bar{F}_i$ and $\Delta_{i-1} \cap \bar{F}_i$ are acyclic, and by (7.37), if Δ_{i-1} is acyclic then so is also Δ_i. Since $\Delta_1 = \bar{F}_1$ is acyclic, it follows by finite induction that so is $\Delta_t = \Delta'$. □

We shall use the notation $\rho(F) = a$ to denote that the k-dimensional face F occurs with the coefficient $a \in \mathbf{Z}$ in the formal linear combination $\rho \in C_k(\Delta)$.

7.7.2. **Theorem.** Let Δ be a shellable d-dimensional complex. Suppose furthermore that {facets F such that $\mathcal{R}(F) = F$} $= \{F_1, F_2, \ldots, F_p\}$, where $\mathcal{R}(F)$ is the restriction operator induced by some shelling. Then

(*i*)
$$H_i(\Delta) \cong \begin{cases} \mathbf{Z}^p, & \text{if } i = d, \\ 0, & \text{if } i \neq d. \end{cases}$$

(*ii*) There are cycles $\rho_1, \rho_2, \ldots, \rho_p \in H_d(\Delta)$ uniquely determined by
$$\rho_k(F_j) = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

(*iii*) $\{\rho_1, \rho_2, \ldots, \rho_p\}$ is a basis of the free group $H_d(\Delta)$.

Notice that for $d = 1$ the theorem states the familiar fact that the cycle space of a connected graph has a basis (unique up to signs) induced by the family of edges lying outside some fixed spanning tree. Notice also that rank $H_d(\Delta) = p = (-1)^d \chi(\Delta)$, by Corollary 7.2.4 (this is a special case of the Euler–Poincaré formula).

For illustration, consider the complex Δ of Example 7.2.1. Taking any shelling (for instance the one mentioned after 7.2.4) one concludes from the theorem that $H_2(\Delta) \cong \mathbf{Z}$ while $H_i(\Delta) = 0$ for all $i \neq 2$. This also follows from the fact (obvious upon inspection) that Δ is homotopy equivalent to the 2–sphere.

Proof. The subcomplex $\Delta' = \Delta - \{F_1, F_2, \ldots, F_p\}$, which by Lemma 7.7.1 is acyclic, differs from Δ only in dimension d. So $C_k(\Delta') = C_k(\Delta)$ for all $k < d$, and consequently $H_k(\Delta) = H_k(\Delta') = 0$ for all $k < d-1$. Also, $B_{d-1}(\Delta') \subseteq B_{d-1}(\Delta) \subseteq Z_{d-1}(\Delta) = Z_{d-1}(\Delta')$ and $H_{d-1}(\Delta') = 0$ imply that $H_{d-1}(\Delta) = 0$. Thus we have proved that $H_i(\Delta) = 0$ for all $i \neq d$.

For $1 \leq k \leq p$ we have that $\partial_d(F_k) \in B_{d-1}(\Delta) \subseteq Z_{d-1}(\Delta) = Z_{d-1}(\Delta')$, and since Δ' is acyclic there exists $\rho'_k \in C_d(\Delta')$ such that $\partial_d(\rho'_k) = \partial_d(F_k)$. Let $\rho_k = F_k - \rho'_k$. By construction $\partial_d(\rho_k) = 0$, so $\rho_k \in Z_d(\Delta) = H_d(\Delta)$, and also $\rho_k(F_j) = \delta_{jk}$ (Kronecker's delta), for $1 \leq j \leq p$. If $\sigma_k \in H_d(\Delta)$ satisfies $\sigma_k(F_j) = \delta_{jk}, 1 \leq j \leq p$, then $\sigma_k - \rho_k \in H_d(\Delta') = 0$, so $\sigma_k = \rho_k$. This proves part (*ii*).

We will now show that $\{\rho_1, \rho_2, \ldots, \rho_p\}$ is a basis of $H_d(\Delta)$. [Remark: One could argue that $H_d(\Delta) = Z_d(\Delta)$ must be free, since it is a subgroup of the free Abelian group $C_d(\Delta)$, but this will of course be a direct consequence of providing a basis.]

Linear independence: Let $\sigma = \sum_{k=1}^{p} a_k \rho_k$, with $a_k \in \mathbf{Z}$. Part (*ii*) shows that $\sigma(F_j) = a_j$, which means that $\sigma = 0$ only if $a_k = 0$ for all $1 \leq k \leq p$.

Generating property: Let $\sigma \in H_d(\Delta)$. Consider the cycle $\tau = \sigma - \sum_{k=1}^{p} \sigma(F_k)\rho_k$. Part (*ii*) shows that $\tau(F_j) = 0$ for all $1 \leq j \leq p$, which means that $\tau \in H_d(\Delta') = 0$. So,

$$\sigma = \sum_{k=1}^{p} \sigma(F_k)\rho_k. \tag{7.38}$$

All claims about the structure of $H_d(\Delta)$ made in parts (*i*) and (*iii*) have now been established. □

The cycles $\rho_1, \ldots, \rho_p$ induced by a shelling are sometimes (e.g. for matroid complexes and geometric lattices) the fundamental cycles of spherical subcomplexes. By this the following is meant. Suppose that a simplicial complex Δ is homeomorphic to the d-sphere. Then $\tilde{H}_d(\Delta) \cong \mathbf{Z}$ and the generator ρ of $\tilde{H}_d(\Delta)$, which is unique up to sign, is a linear combination in which every facet of Δ occurs with coefficient $+1$ or -1. This generator ρ is called the *fundamental cycle* of the spherical complex Δ.

7.8. Homology of matroids

As a direct consequence of Theorem 7.7.2 and our work in Sections 7.3 and 7.4 (see particularly 7.3.3, 7.4.3 and 7.4.7) we obtain the following two results.

7.8.1. **Theorem.** Let $\Delta = IN(M)$ be the complex of independent sets in a matroid M of rank r. Then

$$\tilde{H}_i(\Delta) \cong \begin{cases} \mathbf{Z}^{\bar{\mu}(M^*)}, & \text{if } i = r-1, \\ 0, & \text{if } i \neq r-1. \end{cases}$$

7.8.2. **Theorem.** Let $\Delta = \overline{BC}_\omega(M)$ be the reduced broken circuit complex of an ordered loopless matroid $M = M(S,\omega)$ of rank r. Then

$$\tilde{H}_i(\Delta) \cong \begin{cases} \mathbf{Z}^{\beta(M)}, & \text{if } i = r-2, \\ 0, & \text{if } i \neq r-2. \end{cases}$$

For illustration, consider the rank 3 matroid M of Example 7.3.5. We have in Section 7.3 computed its Tutte polynomial $T_M(x,y) = x^3 + 2x^2 + y^2 + 2xy + x + y$, whose values $\bar{\mu}(M^*) = T_M(0,1) = 2$ and $\beta(M) = \frac{\partial T_M}{\partial x}(0,0) = 1$ should be checked against the topology of the complexes $IN(M)$ and $\overline{BC}_\omega(M)$. These complexes are depicted in Figures 3 and 4.

Theorem 7.8.2 and Proposition 7.4.8 together imply a curious topological duality for reduced broken circuit complexes that seems to lack a systematic explanation. For any matroid $M = M(S)$ without loops or isthmuses, and for any orderings ω, ω' of its ground set S:

$$\tilde{H}_i(\overline{BC}_\omega(M)) \cong \tilde{H}_{|S|-i-4}(\overline{BC}_{\omega'}(M^*)), \tag{7.39}$$

for all $i \in \mathbf{Z}$.

We will now describe a basis for the homology of matroid complexes $IN(M)$. It follows from 7.3.3 and 7.7.2 that such a basis is implicitly determined by any ordering ω of the ground set. What we seek here is a simple explicit description of these bases directly in terms of matroid structure.

Let $M(S,\omega)$ be an ordered matroid of rank r with no isthmus. For each basis B of M construct a simplicial complex $\sum_{B,\omega}$ as follows:

For each $b \in B$ let $\varphi(b) \notin B$ be the least element of the basic bond $bo(B,b) - b$ (which is nonempty since b is not an isthmus), and define elements p_i by $\varphi(B) = [p_1, p_2, \ldots, p_k]$ with $p_1 < \ldots < p_k$. Next, let $A_i = \{p_i\} \cup \varphi^{-1}(p_i)$ for $1 \le i \le k$. The sets A_i are the blocks of a partition of the set $\varphi(B) \cup B$. Finally, let $\sum_{B,\omega} = \{F \subseteq B \cup \varphi(B) : A_i \not\subseteq F \text{ for all } 1 \le i \le k\}$.

7.8.3. **Proposition.** (*i*) $B \in \sum_{B,\omega} \subseteq IN(M)$,
(*ii*) $\sum_{B,\omega}$ is homeomorphic to the $(r-1)$-dimensional sphere.

Proof. As a simplicial complex $D(A) = \{E \subseteq A : E \neq A\}$, for $A \neq \emptyset$, is the boundary of an $(|A|-1)$-simplex, and hence topologically an $(|A|-2)$-sphere. We have that (writing

$\sum_B = \sum_{B,\omega}$ for simplicity)

(7.40) $$\sum_B = D(A_1) * D(A_2) * \ldots * D(A_k),$$

and $|A_i| \geq 2$ for $1 \leq i \leq k$, where the asterisque denotes simplicial join. [The *join* of two simplicial complexes Δ_1 and Δ_2 is defined by $\Delta_1 * \Delta_2 = \{F_1 \cup F_2 : F_1 \in \Delta_1 \text{ and } F_2 \in \Delta_2\}$.] From (7.40) it follows that $B \in \sum_B$ (delete p_i from A_i for all $1 \leq i \leq k$), and hence that $\dim \sum_B = r - 1$. It is well-known in topology that the join of two simplicial spheres is homeomorphic to a sphere (see e.g. Munkres (1984), p. 370), so part (*ii*) follows.

The argument for part (*i*) hinges on the following technical observation:

(7.41) $$1 \leq i < j \leq k \Longrightarrow ci\ (B, p_i) \cap A_j = \emptyset,$$

which follows from the definition of A_j using Lemma 7.3.1.

We will prove by induction on $|F \cap \varphi(B)|$ that every facet F of $\sum_B$ is a basis of M. Clearly, $|F \cap \varphi(B)| = 0$ if and only if $F = B$. Suppose that $F \cap \varphi(B) \neq \emptyset$ and that i is *minimal* such that $p_i \in F \cap \varphi(B)$. There is a unique $b_i \in A_i \cap B$ such that $b_i \notin F$. Let $F' = (F - p_i) \cup b_i$. Then F' is also a facet of $\sum_B$ and $|F' \cap \varphi(B)| = |F \cap \varphi(B)| - 1$, so by the induction hypothesis F' is a basis. It follows from (7.41) that $ci(B, p_i) = ci\ (F', p_i)$, and from $\varphi(b_i) = p_i$ that $b_i \in ci\ (B, p_i)$. Hence, $b_i \in ci\ (F', p_i)$ and we conclude using Lemma 7.3.1 that $F = (F' - b_i) \cup p_i$ is a basis. ◻

For each basis B of M let $\sigma_{B,\omega}$ denote the fundamental cycle of the spherical complex $\sum_{B,\omega}$. There is also the explicit expression (cf. (7.40))

(7.42) $$\sigma_{B,\omega} = \sum_{i_1=0}^{e_1} \cdots \sum_{i_k=0}^{e_k} (-1)^{i_1+\ldots i_k}\ (A_1 - a^1_{i_1}) \cup \ldots \cup (A_k - a^k_{i_k}),$$

where $A_j = [a^j_0, a^j_1, \ldots, a^j_{e_j}]$ is listed increasingly in the ω-ordering, for $1 \leq j \leq k$. Since $\sum_{B,\omega}$ is a full-dimensional subcomplex of $IN(M)$ we have that $\sigma_{B,\omega} \in H_{r-1}(IN(M))$. To remove the sign ambiguity we could demand that $\sigma_{B,\omega}(B) = 1$. [*Remark*: Readers unhappy with the reliance on topology for the derivation of $\sigma_{B,\omega}$ can take (7.42) as its definition and check by direct computation that $\partial_{r-1}(\sigma_{B,\omega}) = 0$ for the simplicial boundary map (7.34).]

For the following result, recall that $i(B)$ denotes the internal activity in a basis B, defined in Section 7.3.

7.8.4. **Theorem.** Let $M = M(S)$ be a matroid of rank r with no isthmus. Then for every ordering ω of S the set of cycles $\{\sigma_{B,\omega} : i(B) = 0\}$ forms a basis for the free Abelian group $H_{r-1}(IN(M))$.

Proof. We apply Theorem 7.7.2 to the ω-lexicographic shelling of $IN(M)$ constructed in Section 7.3. In view of 7.7.2 (*ii*) and (7.9) all that needs to be checked is that if $IP(B) = B$ (equivalently: $i(B) = 0$) and $F \in \sum_{B,\omega}$ for some basis $F \neq B$ then $IP(F) \neq F$.

Suppose then that $IP(B) = B, F \neq B$ and $F \in \sum_{B,\omega}$. Let $F = F_j = (B - \{b_1, \ldots, b_j\}) \cup \{\varphi(b_1), \ldots, \varphi(b_j)\}$, where $b_1, \ldots, b_j \in B$ and $\varphi(b_j) < \ldots < \varphi(b_1)$. We will prove by induction on $j \geq 1$ that $\varphi(b_j)$ is internally active in F_j. For $j = 1$ this is clear, since $\varphi(b_1)$ is the least element of $bo\ (B, b_1) - b_1$ (by definition), $\varphi(b_1) < b_1$ (since $b_1 \in IP(B)$)

and $bo\,(B,b_1) = bo\,(F_1,\varphi(b_1))$. For general $j > 1$, suppose to the contrary that there exists $q \in bo\,(F_j,\varphi(b_j))$ such that $q < \varphi(b_j)$. One can then make the following observations:

(*i*) $q \notin B$ and $q \notin F_{j-1}$,
(*ii*) $b_j \notin ci\,(B,q)$,
(*iii*) $b_j \in ci\,(F_{j-1},q)$,
(*iv*) $\varphi(b_i) \in ci\,(F_{j-1},q)$, for some $1 \le i \le j-1$.

For (*i*) use that $q \notin F_j$ (by definition) and that $q < \varphi(b_j) \le \varphi(b_i) < b_i$ for $1 \le i \le j$, with the last inequality following from the fact that b_i is internally passive in B. Observation (*ii*) is equivalent to $q \notin bo\,(B,b_j)$, which is clear since by definition $\varphi(b_j)$ is minimal in $bo\,(B,b_j) - b_j$ and $q < \varphi(b_j) < b_j$. Similarly, (*iii*) is equivalent to $q \in bo\,(F_{j-1},b_j)$, which follows since $bo\,(F_j,\varphi(b_j)) = bo\,(F_{j-1},b_j)$. Finally, if (*iv*) were false then $ci\,(F_{j-1},q) \subseteq B \cup q$, hence $ci\,(F_{j-1},q) = ci\,(B,q)$ which would contradict (*ii*) and (*iii*).

Now, choose i *maximal* so that $\varphi(b_i) \in ci\,(F_{j-1},q)$. Then $ci\,(F_{j-1},q) \subseteq F_i \cup q$, hence $ci\,(F_{j-1},q) = ci\,(F_i,q)$, and we have that $\varphi(b_i) \in ci\,(F_i,q)$, or equivalently $q \in bo\,(F_i,\varphi(b_i))$. However, since $q < \varphi(b_i)$ this contradicts the induction hypothesis, which says that $\varphi(b_i)$ is internally active in F_i. We conclude that the existence of such an element q is impossible, and the induction step is complete.

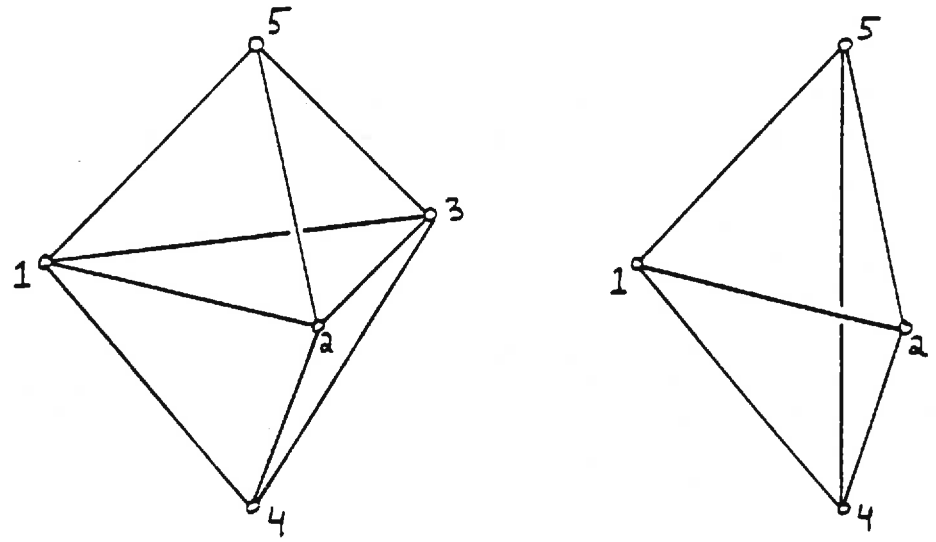
We have shown that $\varphi(b_j)$ is internally active in $F_j = F$, hence $IP(F) \ne F$. □

7.8.5. **Corollary.** $IN(M) = \cup \sum_{B,\omega}$, with union over all bases B of zero internal activity.

Proof. Let B' be an arbitrary basis. From $\sigma_{B',\omega} = \sum_{i(B)=0} n_B \sigma_{B,\omega}$ and $\sigma_{B',\omega}(B') = 1$ it follows that $\sigma_{B,\omega}(B') = \pm 1$, or equivalently $B' \in \sum_{B,\omega}$, for some B such that $i(B) = 0$. □

In summary, we have shown that every isthmus-free matroid complex $IN(M)$ is the union of $\tilde{\mu}(M^*)$ spherical subcomplexes whose fundamental cycles give a basis for homology. Observe that a basis B of M satisfies $i(B) = 0$ if and only if $S - B$ is an *nbc*-basis of the orthogonal matroid M^*. Hence, the broken circuit complex $BC_\omega(M^*)$ plays a role for the homology of $IN(M)$ similar to that played by $BC_\omega(M)$ for $\tilde{H}_{r-2}(\bar{L})$, cf. Theorem 7.9.3.

7.8.6. **Example.** Let M be the matroid of Example 7.3.5. A picture of the complex $IN(M)$ appears in Figure 3. There are two bases of zero internal activity: 235 and 245. For 235 we find that $\varphi(2) = \varphi(3) = 1$ and $\varphi(5) = 4$, so $\sum_{235} = D(123){*}D(45)$ in the notation of (7.40). For 245 we have $\varphi(2) = \varphi(4) = \varphi(5) = 1$, so $\sum_{245} = D(1245)$. These spherical complexes are shown in Figure 7, which should be compared to Figure 3.



(a) Σ_{235} (b) Σ_{245}

Figure 7.

7.9. Homology of geometric lattices

The fact that there are interesting homological aspects of matroid theory was first made clear by the following result of Folkman (1966). Here $\tilde{\mu}(L) = (-1)^r \mu(L) > 0$ is the unsigned Möbuis function value $\mu(L) = \mu(\hat{0}, \hat{1})$.

7.9.1. **Theorem.** Let $\Delta = \Delta(\bar{L})$ be the order complex of a geometric lattice L of rank r. Then

$$\tilde{H}_i(\Delta) \cong \begin{cases} \mathbf{Z}^{\tilde{\mu}(L)}, & \text{if } i = r-2, \\ 0, & \text{if } i \neq r-2. \end{cases}$$

Proof. Follows directly from 7.6.3, 7.6.4 and 7.7.2. □

This section will be devoted to a more detailed study of the homology of geometric lattices. First a basis for the non-vanishing order homology group $\tilde{H}_{r-2}(\Delta(\bar{L}))$ will be described, then a larger object, the *Whitney homology algebra*, $H^W(L)$, will be constructed.

In the following, let (L, ω) be an atom-ordered geometric lattice of rank r. For each nonempty independent set of atoms $A = [a_1, a_2, \ldots, a_k], a_1 < a_2 < \ldots < a_k$, and each permutation $\pi \in S_k$, let

(7.43) $$\mathbf{c}_{A,\pi} : a_{\pi(1)} < (a_{\pi(1)} \vee a_{\pi(2)}) < \ldots < (a_{\pi(1)} \vee a_{\pi(2)} \vee \ldots \vee a_{\pi(k-1)}).$$

Hence, $\mathbf{c}_{A,\pi}$ is a maximal chain in the open interval $(\hat{0}, \bar{A})$ in L (if $k = 1$ this is the empty chain). Furthermore, let

(7.44) $$\rho_A = \sum_{\pi \in S_k} \operatorname{sign}(\pi) \mathbf{c}_{A,\pi}.$$

If one element, say $a_{\pi(1)} \vee \ldots \vee a_{\pi(j)}$, is removed from $\mathbf{c}_{A,\pi}$, then the resulting subchain is contained in exactly one other chain $\mathbf{c}_{A,\pi'}$, namely for the permutation π' which differs from π by transposition of the values of j and $j+1$. Then sign $(\pi') = -\operatorname{sign}(\pi)$, so $\partial_{k-2}(\rho_A) = 0$ for the simplicial boundary operator ∂ defined by (7.34). We have proved the following.

7.9.2. **Lemma.** If $A \neq \emptyset$ is independent and $\bar{A} = x$, then $\rho_A \in \tilde{H}_{r(x)-2}(\hat{0}, x)$.

Here and in the sequel we write simply $\tilde{H}_i(\hat{0}, x)$ for the order homology of the open interval $(\hat{0}, x) = \{z \in L : \hat{0} < z < x\}$, instead of the needlessly pedantic $\tilde{H}_i(\Delta((\hat{0}, x)))$.

The *elementary cycles* ρ_A provide non-zero homology representatives corresponding to all independent sets A. It is now easy to describe a basis for homology. Recall the notation nbc $(x) = \{A \in BC_\omega(L) : \bar{A} = x\}$, for $x \in L$.

7.9.3. **Theorem.** Let $x \in L - \{\hat{0}\}$. Then the elementary cycles $\{\rho_A : A \in \mathbf{nbc}(x)\}$ form a basis for the free Abelian group $\tilde{H}_{r(x)-2}(\hat{0}, x)$.

Proof. We have earlier concluded that $\mathbf{nbc}(x)$ is the family of *nbc*-bases of the geometric lattice $[\hat{0}, x]$ with the induced atom-ordering (see the proof of Proposition 7.4.5). Hence the present proof reduces to the case $x = \hat{1}$, i.e., we must show that the *nbc*-bases of L induce a basis of $\tilde{H}_{r-2}(\bar{L})$.

Using (7.29) and Theorem 7.7.2 all that needs to be checked is that if B is an *nbc*-basis then ρ_B contains exactly one chain $\mathbf{c}_{B,\pi}$ with decreasing label. But this was already shown in (7.33). □

It follows that for every geometric lattice L the order complex $\Delta(\bar{L})$ is the union of $\tilde{\mu}(L)$ spherical subcomplexes whose fundamental cycles give a basis for homology, cf. Exercise 7.9.2.

7.9.4. **Example.** Let L be the geometric lattice considered in Examples 7.3.5, 7.4.4 and 7.6.1, with the natural ordering $\mathbf{1} < \mathbf{2} < \mathbf{3} < \mathbf{4} < \mathbf{5}$ of its atoms, which we here print in boldface to distinguish them from integers.

If x is any atom of L, then $\tilde{H}_{-1}(\hat{0}, x) = \tilde{H}_{-1}(\{\emptyset\}) = \mathbf{Z}\emptyset$ and $\rho_{\{x\}} = \emptyset$.

If $x = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$, then $\tilde{H}_0(\hat{0}, x) = \{c_1 \cdot \mathbf{1} + c_2 \cdot \mathbf{2} + c_3 \cdot \mathbf{3} : c_1 + c_2 + c_3 = 0\}$ and $\rho_{\{1,2\}} = \mathbf{1} - \mathbf{2}, \rho_{\{1,3\}} = \mathbf{1} - \mathbf{3}$.

Finally, let $x = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\} = \hat{1}$, and let $B_1, B_2, \ldots, B_8$ be the bases of L indexed as in Example 7.3.5. The elementary cycles $\rho_i = \rho_{B_i}$ are better expressed by a picture than in algebraic notation, for example see ρ_8 in Figure 8.

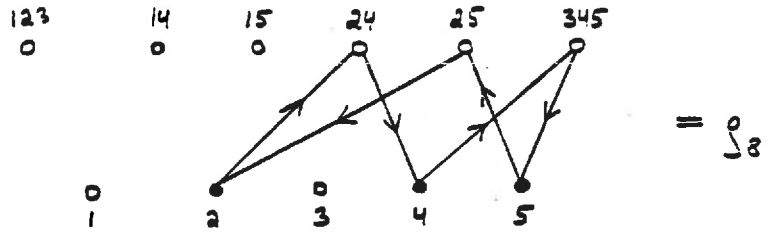


Figure 8.

The *nbc*-bases of L are B_1, B_2, B_3 and B_4, so $\{\rho_1, \rho_2, \rho_3, \rho_4\}$ is a basis of $\tilde{H}_1(\bar{L})$. To express another cycle in terms of this basis we need only check its coefficients on the chains with decreasing labels (by the general principle (7.38)). The four chains with decreasing labels in our example are shown in Figure 9, with the number of the corresponding *nbc*-basis indicated in parentheses.

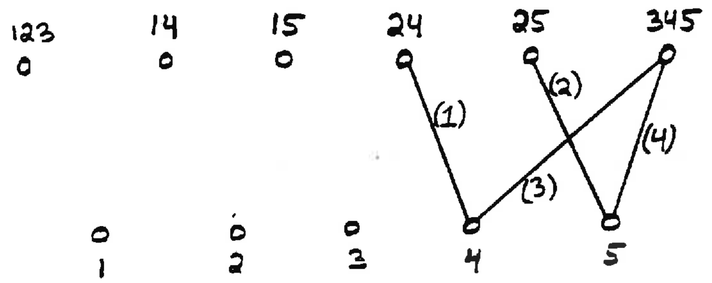


Figure 9.

From the two figures we immediately read off: $\rho_8 = \rho_1 - \rho_2 - \rho_3 + \rho_4$, and similarly, $\rho_5 = -\rho_3 + \rho_4$, $\rho_6 = -\rho_1 + \rho_3$ and $\rho_7 = -\rho_2 + \rho_4$.

The following identity for elementary cycles will later be of use.

7.9.5. **Lemma.** If $C = \{x_1, \ldots, x_k\}$ is a circuit, $x_1 < \ldots < x_k$, then

$$\sum_{i=1}^{k} (-1)^i \rho_{C-x_i} = 0.$$

Proof. Each chain $x_{\pi(1)} < (x_{\pi(1)} \vee x_{\pi(2)}) < \ldots < (x_{\pi(1)} \vee \ldots \vee x_{\pi(k-2)})$, for given $\pi \in S_k$, appears twice in the expansion of the given sum. Once in the term $\rho_{C-x_{\pi(k-1)}}$ and once, with the opposite sign, in the term $\rho_{C-x_{\pi(k)}}$. □

Let L be a geometric lattice of rank r. Introduce a symbol "$\rho_\emptyset$" (the elementary cycle corresponding to the empty set) and formally let $\tilde{H}_{-2}(\hat{0},\hat{0}) = \mathbf{Z}\rho_\emptyset$. Then for each $x \in L$ we have a free Abelian group $\tilde{H}_{r(x)-2}(\hat{0},x)$ of rank $(-1)^{r(x)}\mu(\hat{0},x)$. We now combine all this homology into one global object:

$$\tilde{H}^W(L) = \oplus_{x\in L}\tilde{H}_{r(x)-2}(\hat{0},x), \tag{7.45}$$

called the *Whitney homology* of L. This algebraic object can also be obtained as the homology of an algebraic chain complex (Exercise 7.9.7), and the name "Whitney" stands to indicate that $\tilde{H}^W(L)$ is a direct sum of pieces

$$\tilde{H}_k^W = \oplus_{r(x)=k}\tilde{H}_{k-2}(\hat{0},x),$$

whose ranks are the *Whitney numbers* of the first kind:

$$\text{rank } \tilde{H}_k^W = \sum_{r(x)=k} (-1)^k \mu(\hat{0},x) = \tilde{w}_k. \tag{7.46}$$

From now on, fix an atom-ordering ω of L. The elementary cycles ρ_A of independent subsets $A \subseteq L^1$ are elements in $\tilde{H}^W(L)$, cf. Lemma 7.9.2, and we will now define a combinatorial multiplication rule for them. If $A, B \in IN(L)$, let

$$\rho_A \cdot \rho_B = \begin{cases} \pm\rho_{A\cup B}, & \text{if } A \cap B = \emptyset \text{ and } A \cup B \in IN(L), \\ 0, & \text{otherwise.} \end{cases} \tag{7.47}$$

The sign of $\rho_{A\cup B}$ is positive if the elements of A followed by the elements of B gives an even permutation of $A \cup B$ (all sets listed increasingly); it is otherwise negative. By counting inversions one finds that

$$\rho_A\rho_B = (-1)^{|A|\cdot|B|}\rho_B\rho_A, \text{ for all } A, B \in IN(L). \tag{7.48}$$

Also, $\rho_A\rho_\emptyset = \rho_\emptyset\rho_A = \rho_A$ for all independent sets A.

We will now show that this multiplication uniquely extends to *all* cycles. The relevant algebraic terms are reviewed at the beginning of Section 7.10.

7.9.6. **Theorem.** (i) Whitney homology $H^W(L)$ has a unique structure of an anticommutative L-graded $\mathbf{Z}$-algebra, with grading (7.45) and with multiplication that specializes to (7.47) on elementary cycles.
(ii) A linear basis for $H^W(L)$ is given by $\{\rho_A : A \in BC_\omega(L)\}$.

Proof. Part (ii) follows from Theorem 7.9.3. The work to be done for part (i) lies entirely in showing that the partial multiplication (7.47) extends so that $H_x \cdot H_y \subseteq H_{x\vee y}$, where $H_x = H_{r(x)-2}(\hat{0}, x)$. This global multiplication is then automatically anticommutative, since by (7.48) it is anticommutative on a basis.

The plan for the following proof is to first construct a more simple–minded algebra, whose multiplication is very manageable, and then show that $H^W(L)$ is a subalgebra.

Let C be the set of all lower chains in $L - \{\hat{0}\}$, i.e., $C = \{\emptyset\} \cup \{x_1 < x_2 < \ldots < x_k : r(x_i) = i, 1 \le i \le k\}$. Let $\mathbf{Z}C = \{\sum_{i=1}^{t} a_i \mathbf{c}_i : a_i \in \mathbf{Z}, \mathbf{c}_i \in C\}$ be the Abelian group freely generated by C. Define the product of basis elements $\mathbf{x} : x_1 < x_2 < \ldots < x_k$ and $\mathbf{y} : y_1 < y_2 < \ldots < y_l$ as follows:

$$\mathbf{x} \cdot \emptyset = \emptyset \cdot \mathbf{x} = \mathbf{x},$$

$$\mathbf{x} \cdot \mathbf{y} = \begin{cases} \sum_\sigma \text{sign}(\sigma) \cdot \mathbf{z}(\sigma, \mathbf{x}, \mathbf{y}), & \text{if } r(x_k \vee y_l) = k + l, \\ 0, & \text{otherwise.} \end{cases} \tag{7.49}$$

The summation in (7.49) is over all (k,l)*–shuffles* σ, i.e., all permutations σ of $\{1, 2, \ldots, k + l\}$ such that $\sigma^{-1}(1) < \ldots < \sigma^{-1}(k)$ and $\sigma^{-1}(k+1) < \ldots < \sigma^{-1}(k+l)$; and $\mathbf{z}(\sigma, \mathbf{x}, \mathbf{y})$ denotes the lower chain $z_{\sigma(1)} < (z_{\sigma(1)} \vee z_{\sigma(2)}) < \ldots < (z_{\sigma(1)} \vee z_{\sigma(2)} \vee \ldots \vee z_{\sigma(k+l)})$, where $(z_1, z_2, \ldots, z_{k+l}) = (x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_l)$.

The mapping $C \times C \longrightarrow \mathbf{Z}C$ thus defined extends by linearity to a multiplication $\mathbf{Z}C \times \mathbf{Z}C \longrightarrow \mathbf{Z}C$ (associativity requires checking), making $\mathbf{Z}C$ into a ring with identity element $\emptyset$.

We have the following situation:

$$H^W(L) = \oplus_{x\in L} H_{r(x)-2}(\hat{0}, x) \xrightarrow{\varphi} \oplus_{x\in L} C_{r(x)-2}(\hat{0}, x) \xrightarrow{\psi} \mathbf{Z}C.$$

Here $C_{r(x)-2}(\hat{0}, x)$ is the chain group in the sense of Section 7.7, i.e., the free Abelian group generated by all maximal chains in the open interval $(\hat{0}, x)$, and φ is the embedding of the subgroup of $(r(x) - 2)$–cycles, for all $x > \hat{0}$. Also, put $C_{-2}(\hat{0}, \hat{0}) = H_{-2}(\hat{0}, \hat{0}) = \mathbf{Z}\rho_\emptyset$. The mapping ψ is defined by sending the basis element $y_1 < y_2 < \ldots < y_{r(x)-1}$ in $C_{r(x)-2}(\hat{0}, x)$ to the corresponding basis element $y_1 < y_2 < \ldots < y_{r(x)-1} < x$ in $\mathbf{Z}C$, with appropriate modification for the degenerate case $x = \hat{0}$. These mappings have the following key properties:

(*i*) φ *is an injection of L–graded Abelian groups* (clear),

(*ii*) ψ *is an isomorphism of Abelian groups* (clear), and

(*iii*) **Z**C *is an anticommutative L– graded algebra under the direct sum decomposition* ψ^{-1} (follows from definition (7.49)).

We conclude that

$$\tau = \psi \circ \varphi : H^W(L) \longrightarrow \mathbf{Z}C$$

is an L-graded embedding of $H^W(L)$ as a subgroup of the L-graded algebra $\mathbf{Z}C$. To finish the proof we will show that

$$(iv) \qquad \tau(\rho_A) \cdot \tau(\rho_B) = \begin{cases} \pm\, \tau(\rho_{A \cup B}), & \text{if } A \cap B = \emptyset \text{ and } A \cup B \in IN(L), \\ 0, & \text{otherwise,} \end{cases}$$

where $A, B \in IN(L)$ *and the rule for signs is as in* (7.47),

$$(v) \qquad H^W(L) \cong Im\ \tau \text{ is multiplicatively closed in } \mathbf{Z}C.$$

If $A \cap B \neq \emptyset$ or if $A \cup B$ is dependent, then $r(\bar{A} \vee \bar{B}) < r(\bar{A}) + r(\bar{B})$, and if $A = [a_1, \ldots, a_k], B = [b_1, \ldots, b_l]$, rule (7.49) shows that $\tau(\rho_A) \cdot \tau(\rho_B) = 0$ in $\mathbf{Z}C$. Otherwise $r(\bar{A} \vee \bar{B}) = r(\bar{A}) + r(\bar{B})$, and if $A = [a_1, \ldots, a_k]$, $B = [b_1, \ldots, b_l]$, rule (7.49) shows that $\tau(\rho_A) \cdot \tau(\rho_B)$ is a linear combination of signed chains $\mathbf{z}(\sigma, \bar{\mathbf{c}}_{A,\pi}, \bar{c}_{B,\pi'})$ where $\sigma \in S_{k+l}/S_k \times S_l$ (the set of (k,l)–shuffles), $(\pi, \pi') \in S_k \times S_l$, and $\bar{\mathbf{c}}_{A,\pi}$ is the chain (7.43) augmented by $a_{\pi(1)} \vee a_{\pi(2)} \vee \ldots \vee a_{\pi(k)} = \bar{A}$. But via the natural bijection $(S_{k+l}/S_k \times S_l) \times (S_k \times S_l) \longleftrightarrow S_{k+l}$ these chains are in sign–compatible bijection with the chains $\bar{\mathbf{c}}_{A \cup B, \nu}$, for $\nu \in S_{k+l}$, that occur in $\tau(\rho_{A \cup B})$. This proves (*iv*).

We know that $\{\tau(\rho_A) : A \in BC_\omega(L)\}$ is a basis for $Im\ \tau$, and (*iv*) has shown that products of basis elements remain in $Im\ \tau$. Hence, (*v*) follows. □

7.10. The Orlik–Solomon algebra

In this section a certain anticommutative L-graded algebra $\mathcal{A}(L)$ will be constructed, for each geometric lattice L. The basic combinatorial properties of $\mathcal{A}(L)$ will be derived with emphasis on its intimate ties to the order homology of L.

We begin with a brief review of definitions and basic facts concerning graded algebras and exterior algebra. More details can be found in algebra books, such as Bourbaki (1970).

Let M be a commutative monoid with identity element e, whose composition we write multiplicatively, and let R be a commutative ring. By an *M–graded R–algebra A* we mean a ring A together with a direct sum decomposition $A = \oplus_{x \in M} A_x$ (as an additive group) such that $A_e \cong R$ and $A_x \cdot A_y \subseteq A_{xy}$. In particular, via the identification $A_e = R$, each A_x is an R–module. An element $a \in A$ is called *homogeneous* if $a \in A_x$ for some $x \in M$, and an ideal is *homogeneous* if it is generated by homogeneous elements. Equivalently, an ideal $I \subseteq A$ is homogeneous if and only if $I = \oplus_{x \in M}(I \cap A_x)$. For any homogeneous ideal I there is a naturally induced structure of M–graded R–algebra on its quotient: $A/I \cong \oplus_{x \in M}(A_x / I \cap A_x)$.

In most cases where graded algebras occur, $M = (\mathbf{N}^r, +)$ and R is a field. We shall mainly use $M = (L, \vee)$, where L is a geometric lattice, and $R = \mathbf{Z}$. The reason for using integer coefficients is to get sharper algebraic statements, in particular to show that no torsion arises. The following arguments would go through with an arbitrary ring R instead of $\mathbf{Z}$, on condition that R–coefficients had been used also in our previous work on homology.

An L-graded algebra $A = \oplus_{x \in L} A_x$, where L is a geometric lattice, will be called *anticommutative* if $a \cdot b = (-1)^{r(x) \cdot r(y)} b \cdot a$ for all $a \in A_x, b \in A_y$.

Let E be a free Abelian group with basis $\{e_1, e_2, \ldots, e_n\}$. The exterior algebra $\Lambda E = \oplus_{k=o}^{n} \Lambda^k E$ is an N-graded Z-algebra with main properties (multiplication denoted by "$\wedge$"):

(1) $x \wedge y = (-1)^{k \cdot l}\ y \wedge x$, for $x \in \Lambda^k E$ and $y \in \Lambda^l E$. In particular, for $x, y \in E = \Lambda^1 E : x \wedge y = -y \wedge x$ (so , $x \wedge x = 0$).
(2) $\Lambda^k E$ is a free Abelian group with basis $\{e_A :$ card $A = k\}$, where $A = [i_1, i_2, \ldots, i_k], 1 \leq i_1 < i_2 < \ldots < i_k \leq n$, and

(7.50) $$e_A = e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}.$$

Consequently, rank $\Lambda^k E = \binom{n}{k}$.
(3) The multiplication of basis elements is (as follows from (1) and (2)) given by:

$$e_A \wedge e_B = \begin{cases} \pm e_{A \cup B}, & \text{if } A \cap B = \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

with the sign determined by the parity of the permutation that brings A followed by B to $A \cup B$ (all sets listed increasingly).

Now, let L be a geometric lattice and let $E(L)$ be the Abelian group freely generated by the set of its atoms $L^1 = \{e_1, e_2, \ldots, e_n\}$. Whenever convenient we may assume given also an atom-ordering $\omega : e_1 < e_2 < \ldots < e_n$. For $A \subseteq L^1$ let $e_A \in \Lambda E(L)$ have the meaning (7.50). Then,

$$\Lambda E(L) = \oplus_{x \in L} \Lambda_x, \text{ where } \Lambda_x = \oplus_{\bar{A} = x} \mathbf{Z} e_A,$$

and this direct sum decomposition gives $\Lambda E(L)$ the structure of an L-graded Z-algebra, since $e_A \wedge e_B \in \mathbf{Z}\ e_{A \cup B} \subseteq \Lambda_{\bar{A} \vee \bar{B}}$.

For each circuit $C = \{x_1, x_2, \ldots, x_k\} \subseteq L^1, x_1 < \ldots < x_k$, let $\partial(e_C) = \sum_{i=1}^{k} (-1)^i\ e_{C - x_i}$, and let I be the two-sided ideal of $\Lambda E(L)$ generated by the elements $\partial(e_C)$ for all circuits C in L. The quotient

(7.51) $$\mathcal{A}(L) = \Lambda E(L)/I$$

is the *Orlik–Solomon* **algebra** of L. Since I is a homogeneous ideal (the generators $\partial(e_C) \in \Lambda_{\bar{C}}$ being homogeneous), $\mathcal{A}(L)$ inherits an L-grading from $\Lambda E(L)$:

(7.52) $$\mathcal{A}(L) = \oplus_{x \in L} \mathcal{A}_x, \text{ where } \mathcal{A}_x = \Lambda_x \ / \ (I \cap \Lambda_x).$$

Denote by $\bar{e}_A$ the class of e_A in $\mathcal{A}(L)$, for subsets $A \subseteq L^1$.

7.10.1. **Lemma.** (*i*) $\bar{e}_A \neq 0$ if and only if A is independent.
(*ii*) $\{\bar{e}_A : A \in BC_\omega(L)\}$ linearly generates $\mathcal{A}(L)$.

Proof. (*i*) Suppose A is dependent, and let $e_i \in C \subseteq A$ for some circuit C. Then $e_A = \pm e_i \wedge \partial(e_C) \wedge e_{A-C} \in I$. The converse, which will not be needed in the sequel, is a consequence of Theorem 7.10.2.
(*ii*) Suppose that $\bar{e}_A \neq 0$, and that A contains the broken circuit $C - x_1$, where $C = \{x_1, \ldots, x_k\}$, $x_1 < \ldots < x_k$. Then A is independent by (*i*), so $x_1 \notin A$. The relation $\partial(e_C) \wedge e_{A-C} \in I$ takes the form

$$\bar{e}_A = \sum_{i=2}^{k} (-1)^i\ \bar{e}_{(A \cup x_1) - x_i},$$

expressing $\bar{e}_A$ as a linear combination of $\bar{e}_B$'s with index set B lexicographically preceding A. If any such B contains a broken circuit, $\bar{e}_B$ can be similarly expanded in terms of lexicographically earlier elements, and so on. When the process eventually stabilizes, all index sets must be free of broken circuits. □

We now come to the main result of this section, which relates the Orlik–Solomon algebra $\mathcal{A}(L)$ to the Whitney homology algebra $H^W(L)$ of Section 7.9. For both objects a family of elements $\bar{e}_A$ resp. ρ_A (elementary cycles) indexed by independent sets A play an important role.

7.10.2. **Theorem.** (*i*) The correspondence $\bar{e}_A \longleftrightarrow \rho_A$, for independent subsets $A \subseteq L^1$, extends to an isomorphism $\mathcal{A}(L) \cong H^W(L)$ of anticommutative L–graded algebras.
(*ii*) $\{\bar{e}_A : A \in BC_\omega(L)\}$ is a linear basis for $\mathcal{A}(L)$.

Proof. Define a surjective linear mapping

$$h : \Lambda E(L) \longrightarrow H^W(L)$$

by letting $h(e_A) = \rho_A$, if A is independent, and $h(e_A) = 0$ otherwise. The rule (7.47) shows that h preserves multiplication, so h is actually a surjective ring–homomorphism. Furthermore, Lemma 7.9.5 shows that $h(\partial(e_C)) = 0$ for every circuit C, so $I \subseteq Ker\ h$. Therefore h induces a surjective ring–homomorphism

$$\bar{h} : \mathcal{A}(L) \longrightarrow H^W(L),$$

such that $\bar{h}(\bar{e}_A) = \rho_A$, for all $A \in IN(L)$. Specializing to $A \in BC_\omega(L)$, and using Theorem 7.9.6 (*ii*) and Lemma 7.10.1 (*ii*), we then deduce the result from the following obvious lemma: If $f : G \longrightarrow H$ is a surjective linear map of Abelian groups sending a generating set in G to a linearly independent set in H, then f is an isomorphism and both sets are bases. □

For $0 \leq k \leq r =$ rank (L), let

$$\mathcal{A}^k(L) = \oplus_{r(x)=k}\, \mathcal{A}_x \cong \Lambda^k E(L)\ /\ (I \cap \Lambda^k E(L)). \tag{7.53}$$

Then $\mathcal{A}(L) = \oplus_{k=0}^{r}\, \mathcal{A}^k(L)$ gives $\mathcal{A}(L)$ structure of an N–graded anticommutative algebra. An expression for the Hilbert–Poincaré polynomial of $\mathcal{A}(L)$ under this coarser grading follows directly from (7.46) and the isomorphism $\mathcal{A}_x \cong \tilde{H}_{r(x)-2}\,(\hat{0}, x)$.

7.10.3. **Corollary.** $\sum_{k=0}^{r}$ (rank $\mathcal{A}^k(L)$) $t^k = \sum_{k=0}^{r} \tilde{w}_k\ t^k = (-t)^r p(L; -\frac{1}{t})$, where $p\ (L;\lambda)$ is the characteristic polynomial of L (defined in Section 7.4).

7.11. Notes and Comments

We end with references to original sources and related remarks. Additional results can be found among the exercises.

§**7.1.** There are several known topological aspects of matroid theory, in addition to the topics treated here (which are mostly of an algebraic nature). We would like to mention: (1) the homotopy theorems of Tutte and Maurer, (2) the topological theory of oriented matroids (particularly the Folkman–Lawrence representation theorem), (3) simplicial matroids, and (4) matroid versions of Sperner's lemma. Expository accounts with further references appear for (1) and (2) in Björner (1990) and for (3) and (4) in White (1987), Chapter 6.

§7.2. The notion of shellability originated in polytope theory in connection with attempts from the mid–1800's on to prove the generalized Euler formula by induction (Grünbaum, 1967). It has been most intensively studied for polytopes and spheres (Danaraj and Klee, 1978), but recently also for many other types of complexes (Björner, 1990). Many of the basic combinatorial properties of a shellable complex, such as 7.2.2 and the role of the h–vector, are due to McMullen (1970).

A numerical characterization of the h–vectors of shellable complexes was given by Stanley (1977). The proper setting for this result is the theory of Cohen–Macaulay complexes (Stanley, 1977, 1983). Proposition 7.2.5 depends only on the most elementary properties of h–vectors (such as non-negativity), and is valid for all Cohen–Macaulay complexes. Part (i) has been generalized to all pure complexes by Stanley (unpublished), and further to all pure multicomplexes by Hibi (1989).

§7.3 – 7.4. The shellability of matroid and broken circuit complexes was first proven by Provan (1977), see Provan and Billera (1980), using the recursive method of "vertex decomposability". The lexicographic method presented here and its close connection to the concepts of internal/external activity and the Tutte polynomial was found by Björner (1979). [*Remark*: This reference is a preprint version of Sections 7.2 – 7.5 of this chapter.]

There are two complementary approaches to Tutte polynomials: the recursive approach based on contraction and deletion (see Chapter 6), and the constructive or generating function approach (of which a glimpse has been given here). The theory of Tutte polynomials was generalized from graphs to matroids by Crapo (1969), to whom 7.3.6 and 7.3.7 are due. A generalization of 7.3.6 appears in Dawson (1981).

The h-vectors and Stanley–Reisner rings of matroid complexes are discussed by Stanley (1977). He proves that such h-vectors are "level sequences", but his conjecture that they are "pure O-sequences" is still open. The result of Exercise 7.3.3. implies that $IN(M)$ is "$(n-h)$-Cohen–Macaulay connected", which Baclawski (1982) shows has interesting consequences for the Betti numbers and canonical module of the Stanley–Reisner ring of $IN(M)$. The h-vector of $IN(M)$ is also studied by Dawson (1984).

Theorem 7.4.6, the key enumerative fact about broken–circuit–free sets, was discovered by Whitney (1932) for graphs and extended to matroids by Rota (1964). The proof given for 7.4.5 – 7.4.6 is from Björner and Ziegler (1987). The broken circuit complex was for the first time considered *as a simplicial complex* by Wilf (1976), for the purpose of studying Whitney numbers (and ultimately the chromatic polynomials of graphs). Many of the basic combinatorial properties of (reduced) broken circuit complexes, such as 7.4.7, are due to Brylawski (1977a). See also Brylawski and Oxley (1980, 1981) and Björner and Ziegler (1987).

§7.5. Inequalities for independence numbers and Whitney numbers of matroids have a sizeable literature. Much work in this area has been motivated by the still open unimodality conjectures (Mason, 1972). For a survey of Whitney numbers see Chapter 8 of White (1987), for independence numbers see Welsh (1976),Dowling (1980), and Mahoney (1985). Applications of Whitney number inequalities to the enumeration of cells in hyperplane arrangements are discussed by Greene and Zaslavsky (1983) and Zaslavsky (1981, 1983).

Matroid inequalities reflect the deeper and more intrinsic question of characterizing the h–vectors of matroid and broken circuit complexes, about which quite little is known. The direct connection to characteristic polynomials (in the case of broken circuit complexes) indicates that these questions are likely to be very difficult.

The material in this section is from Björner (1979, 1980b). The $c = 2$ and $c = 3$ cases of 7.5.2 were independently found by Purdy (1982), and the $c = 3$ cases of 7.5.6 and

7.5.8, i.e., (7.24) and (7.25) – (7.26), are due to Dowling and Wilson (1974) and Brylawski (1977b), respectively. Heron (1972) had earlier found the inequalities 7.5.6, but did not characterize the case of equality. Formula (7.22) for connected matroids is due to G. Dinolt and U. Murty, see Welsh (1976), p. 299.

§7.6. Theorem 7.6.3 is from Björner (1980a), and represents a special case of the method of lexicographic shellability for posets. The edge–labelings (7.27) had earlier been used by Stanley (1974) to prove a more general version of 7.6.4 (*ii*), see Exercise 7.6.5. The bijection (7.30) also appears in that paper.

The lexicographic shellability of geometric lattices has been extended to related more general posets by Wachs and Walker (1986) and Laurent and Deza (1988). Also, the particular shellings constructed in Exercise 7.6.6 (c) have been generalized from modular geometric lattices to all Tits buildings in Björner (1984).

The *facet graph* of a pure simplicial complex Δ has as its vertices the facets of Δ and as edges the pairs of facets that differ in only one element. Facet graphs of matroid complexes $IN(M)$ were studied by Maurer (1973), who obtained a characterization of this class of graphs. The facet graphs of geometric lattice complexes $\Delta(\bar{L})$ have been studied by Abels (1989, 1990). As discussed by Björner (1984) and Abels (1990) there are several structural similarities between geometric lattice complexes on the one hand and Tits buildings on the other. This is of course not surprising, since both structures generalize (in different directions) the properties of the subspace lattice of a finite-dimensional vector space. The similarities concern the role played by "frames" (Boolean sublattices and apartments, respectively) and metric properties of the facet graphs.

§7.7. The material discussed is in principle well known. Part (*i*) of 7.7.2 can be sharpened to the statement that Δ has the homotopy type of a wedge of p copies of the d–sphere.

For more about simplicial homology, and algebraic topology generally, see e.g. Munkres (1984). Björner (1990) surveys applications to combinatorics, including more details about shellable complexes.

§7.8. Theorems 7.8.1 and 7.8.2 appear to be due to Stanley, they are certainly implicit in Stanley (1977). The spheres $\sum_{B,\omega}$ were considered by Cordovil (1985) for other purposes (generalizations of Sperner's lemma). Proposition 7.8.3 is due to him. Theorem 7.8.4 appears to be new.

§7.9. Theorem 7.9.1 is due to Folkman (1966), and the basis results 7.9.3 and 7.9.6 (*ii*) to Björner (1982). Whitney homology $H^W(L)$ was introduced via sheaf cohomology by Baclawski (1975). The exact relationship of Whitney homology to order homology, taken here as the definition of $H^W(L)$, was analyzed by Björner (1982) and Orlik and Solomon (1980), see Exercise 7.9.7. Our construction of the multiplicative structure on $H^W(L)$ is based on ideas of Orlik and Solomon (1980).

A surprising connection between the homology of finite partition lattices and free Lie algebras has been discovered by Barcelo (1988). She proves a direct correspondence between the homology basis of the broken circuit complex and the free Lie algebra basis of Lyndon words.

§7.10. The definition of $\mathcal{A}(L)$ and the isomorphism 7.10.2 (*i*) are due to Orlik and Solomon (1980). The basis result 7.10.2 (*ii*) is immediately implied by 7.10.2 (*i*) and 7.9.6 (*ii*). It was independently discovered by Jambu and Leborgne (1986), see also Jambu and Terao (1989), Gel'fand and Zelevinskii (1986), and Zelevinskii (1988).

The algebra $\mathcal{A}(L)$ was introduced by Orlik and Solomon to give a combinatorial presentation of the cohomology ring of the complement in $\mathbf{C}^d$ to a finite union of central $(d-1)$–planes. It has also found use in the work of Gel'fand, Zelevinskii, and coworkers, on hypergeometric functions. See Orlik (1989) for an expository account and further references.

A finite group acting on a geometric lattice L of rank r has an induced action on the homology $\tilde{H}_{r-2}(\bar{L})$. More generally, for each $J \subseteq \{1,2,\ldots,r-1\}$ the group has a representation on the homology of the rank–selected subposet $L^J = \{x \in L : r(x) \in J\}$, and for each $0 \le k \le r$ a representation of degree $\tilde{w}_k$ on the graded component $\mathcal{A}^k(L)$ of the Orlik–Solomon algebra $\mathcal{A}(L)$. Such representations were first systematically studied by Stanley (1982) and Orlik and Solomon (1980), respectively. Later papers include Barcelo (1988), Barcelo and Bergeron (1988), Calderbank, Hanlon and Robinson (1986), Hanlon (1984, 1989), Lehrer (1987), Lehrer and Solomon (1986), and Rotman (1985).

Exercises

Problems whose solution is unknown to the author are denoted by an asterisque.

7.2.1. Let Δ be a cone with apex $v \in V$, and let $\Delta' = \{F \in \Delta : v \notin F\}$. Show that Δ is shellable if and only if Δ' is shellable, and if so their shelling polynomials satisfy $h_\Delta(x) = x \cdot h_{\Delta'}(x)$.

7.2.2. This exercise refers to Proposition 7.2.5.
(a) Prove part (i).
(b) Show that $h_i \geq 1$ implies $h_{i-1} \geq 1$ and $h_i \geq 2$ implies $h_{i-1} \geq 2$, for $1 \leq i \leq r$.
(c) Prove part (ii).
(d) Show that for $r \geq 3$:

$$f_1 < f_2 < \ldots < f_{[r/2]} \leq f_{[r/2]+1},$$

where if r is even the last inequality presupposes that $h_{[r/2]+1} \geq 1$. The last inequality is strict if r is odd or if $h_2 \geq 2$.

7.2.3. Say that a pure simplicial complex Δ is *strongly connected* if every pair of facets F and G are connected by a sequence of facets $F = F_0, F_1, \ldots, F_k = G$ such that codim $(F_{i-1} \cap F_i) = 1$, for $1 \leq i \leq k$. Show the following:
(a) Every shellable complex is strongly connected.
(b) If Δ is $(r-1)$–dimensional, strongly connected and has v vertices, then

$$f_k \geq \binom{r}{k} + (v-r)\binom{r-1}{k-1},$$

for $0 \leq k \leq r$.
(c) Equality holds in (b) for all k (or, equivalently, for $k = r$) if and only if Δ is shellable and $h_2 = 0$.

7.3.1. Show that a simplicial complex Δ on vertex set V is a matroid complex if and only if the induced subcomplex $\Delta_A = \{F \in \Delta : F \subseteq A\}$ is pure for all subsets $A \subseteq V$.

7.3.2. Let $M(S,\omega)$ be an ordered matroid, and let $\mathcal{F} = \{IP(B) : B \text{ a basis of } M\}$.
(a) Show that if $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$. [Dawson (1984)].
(b) Deduce using Lemma 7.2.6 and (7.9) that $\mathcal{F}$ is a greedoid. [See Section 8.2 for the definition.]
(c) Show that $\mathcal{F}$ has the interval property. [M. Purtill, 1986 (unpublished)].
(d)* Characterize those interval greedoids that arise from an ordered matroid in this way. What can be said about their f–vectors (i.e., the h–vectors of matroid complexes)?

7.3.3. Let $M(S)$ be a rank r matroid of size n, and define h to be the maximal size of any hyperplane. Show that for any subset $A \subseteq S$ such that $|A| < n - h$, the subcomplex of $IN(M)$ induced on $S - A$ is shellable and of dimension $r - 1$.

7.3.4. Consider the complete graph K_n on labeled vertices $1,2,\ldots,n$. Let c_k be the number of connected spanning subgraphs of K_n with k edges, and define $C_n(t) = \Sigma_k c_k t^k$.
(a) Show that $C_n(t) = t^{n-1} \cdot T_{K_n}(1, 1+t)$.
[*Hint*: Use (7.12) and Proposition 7.2.3.]
Now, let $\mathcal{T}_n$ be the set of all spanning trees of K_n. Define an *inversion* of a tree $T \in \mathcal{T}_n$ to be an ordered pair (i,j) of vertices such that $i > j > 1$ and such that the unique path from 1 to j passes through i. Let $inv(T)$ be the number of inversions in T, and define $I_n(t) = \Sigma_{T\in\mathcal{T}_n} t^{inv(T)}$.
(b) Show that $C_n(t) = t^{n-1} \cdot I_n(1+t)$.
[Gessel and Wang (1979)]
(c) Conclude that the number of trees in $\mathcal{T}_n$ with k inversions equals the number of trees in $\mathcal{T}_n$ with k externally active edges (with respect to a fixed arbitrary ordering of the edges of K_n).
(d) Prove part (c) directly by constructing a bijection between the two classes of trees.[Beissinger (1982)]

7.4.1. Show that every matroid complex is a reduced broken circuit complex, i.e., given a matroid M construct an ordered matroid (M',ω) such that $IN(M) = \overline{BC}_\omega(M')$. [Brylawski (1977a)]

7.4.2.* (a) Characterize (reduced) broken circuit complexes. [Brylawski (1977a)]
(b) Characterize the f-vectors (or, h-vectors) of broken circuit complexes. [Wilf (1976) and others]

7.4.3. Let $M(S)$ be a simple matroid of rank r and size n with characteristic polynomial $p(M;\lambda)$. Let k be a positive integer and suppose that $k+1$ colors are available, one of which is blue. Show that $k^{n-r}(-1)^r p(M;-k)$ equals the number of $(k+1)$-colorings of S which have no blue broken circuit. (Broken circuits are defined with respect to some fixed but arbitrary ordering of S.) [Wilf (1977), Brylawski (1977a)]

7.4.4. (a) Let (L,ω) be an atom-ordered geometric lattice, and let A be an ω-initial segment of L^1. Show that $BC_\omega(L) = \Delta_A * \Delta_B$, where Δ_A and Δ_B are the subcomplexes induced on the complementary subsets A and $B = L^1 - A$, if and only if A is a modular flat. [Brylawski (1977a). The *join* of two simplicial complexes Δ_1 and Δ_2 is defined by $\Delta_1 * \Delta_2 = \{F_1 \cup F_2 : F_1 \in \Delta_1 \text{ and } F_2 \in \Delta_2\}$.]
(b) Deduce using (a) that the characteristic polynomial $p([\hat{0}, x];\lambda)$ of a modular flat $x \in L$ divides $p(L;\lambda)$. [Stanley (1971)]

7.4.5. Let L be a supersolvable geometric lattice with M-chain $\hat{0} = m_0 < m_1 < \ldots < m_r = \hat{1}$. For $1 \le i \le r$, let $A_i = \{p \in L^1 : p \le m_i, p \not\le m_{i-1}\}$ and $a_i = |A_i|$. Show the following:
(a) Let ω be an atom-ordering such that all of A_i comes before all of A_{i+1}, for $1 \le i \le r-1$. Then $BC_\omega(L) = \Delta_{A_1} * \Delta_{A_2} * \ldots * \Delta_{A_r}$.
(b) Each subcomplex Δ_{A_i} in (a) is 0-dimensional.
(c) $p(L;\lambda) = (\lambda - a_1)(\lambda - a_2)\ldots(\lambda - a_r)$.
[A geometric lattice is called *supersolvable* if it has an M-chain, meaning a maximal chain of modular elements. This concept is due to Stanley (1972), who also proved (c). Parts (a) and (b) follow from Brylawski (1977a).]

7.4.6. Show that the following conditions on a geometric lattice L of rank r are equivalent:
(a) L is supersolvable.

(b) For some atom–ordering ω, the broken circuit complex $BC_\omega(L)$ is a multiple join of 0–dimensional subcomplexes.
(c) For some ω there exists a partition $L^1 = A_1 \cup A_2 \cup \ldots \cup A_r$, such that $|B \cap A_i| = 1$ for all *nbc*–bases B and all $1 \le i \le r$.
(d) For some ω, the 1–skeleton of $BC_\omega(L)$ is a complete r–partite graph.
(e) For some ω, the 1–skeleton of $BC_\omega(L)$ is an r–partite graph.
(f) For some ω, the inclusion–wise minimal broken circuits all have size 2.
(g) There exists a partition $L^1 = A_1 \cup A_2 \cup \ldots \cup A_r$ such that for any two distinct $x, y \in A_i$ there is a $z \in A_j$, for some $j < i$, such that $\{x, y, z\}$ is a circuit.
[Björner and Ziegler (1987)]

7.4.7. Show that a supersolvable geometric lattice is determined by its 3–truncation, i.e., it can be reconstructed from its point–line incidences.
[Halsey (1987), Björner and Ziegler (1987)]

7.4.8. For every geometric lattice L, certain families of subsets of L^1 (the set of atoms), called *neat base–families*, are recursively defined as follows:
(*i*) If rank $L = 1$, then $\{\{\hat{1}\}\}$ is a neat base–family.
(*ii*) If rank $L > 1$, then pick an arbitrary atom $p \in L^1$ and for each hyperplane $h \in L^{r-1}$ such that $h \not\geq p$ let $\mathcal{B}_h$ be a neat base–family in $[\hat{0}, h]$. Then $\mathcal{B} = \{A \cup p : A \in \mathcal{B}_h, h \not\geq p\}$ is a neat base–family in L.
Show the following:
(a) Every member of a neat base–family is a basis of L.
(b) Every neat base–family has $\tilde{\mu}(L)$ members.
(c) The facets of any broken circuit complex $BC_\omega(L)$ is a neat base–family, but not conversely.
[Björner (1982)]

7.4.9. Let L be a geometric lattice and $\pi : L - \hat{0} \longrightarrow L^1$ a map, such that $\pi(x) \le x$ and $\pi(x) \le y \le x$ implies $\pi(y) = \pi(x)$, for all $x, y \in L - \hat{0}$. A subset $A \subseteq L^1$ is called *rooted* if $\pi(\overline{B}) \in B$ for all nonempty subsets $B \subseteq A$. The collection of rooted sets for L and π, denoted $RC_\pi(L)$, is called a *rooted complex*.
Show the following:
(a) $RC_\pi(L)$ is a simplicial complex.
(b) $RC_\pi(L)$ is pure of dimension rank $L - 1$.
(c) All members of $RC_\pi(L)$ are independent. [I.e., $RC_\pi(L)$ is a full-dimensional subcomplex of $IN(L)$.]
(d) $RC_\pi(L)$ is a cone with apex $\pi(\hat{1})$.
(e) Define *π–broken circuits* as subsets of the form $C - \pi(\overline{C})$, for circuits C such that $\pi(\overline{C}) \in C$. Then $RC_\pi(L)$ consists precisely of those subsets of L^1 that do not contain any π–broken circuit.
(f) For $x \in L - \hat{0}$, put $\mathbf{rc}\ (x) = \{A \in RC_\pi(L) : \overline{A} = x\}$. Then $\mathbf{rc}(x)$ is a neat base–family in $[\hat{0}, x]$.
(g) For all $x \in L$, $|\mathbf{rc}\ (x)| = (-1)^{r(x)}\mu(\hat{0}, x)$.
(h) The face numbers of $RC_\pi(L)$ coincide with the Whitney numbers of the first kind of L.
(i)* Is $RC_\pi(L)$ shellable?
[Björner and Ziegler (1987)]

7.4.10 Show that every broken circuit complex $BC_\omega(M)$ is a rooted complex $RC_\pi(L)$, but not conversely.
[Björner and Ziegler (1987)]

7.5.1. For a loopless matroid M of rank r, show that:
(a) $T_M(x,0) = x^r + a_1x^{r-1} + \ldots + a_kx^k$, with $a_k \neq 0$, if and only if M is the direct product of k connected simple matroids.
(b) $T_M(x,1) = x^r + b_1x^{r-1} + \ldots + b_kx^k$, with $b_k \neq 0$, if and only if M has exactly k isthmuses.

7.5.2. This exercise refers to Proposition 7.5.1.
(a) If M has exactly p isthmuses, show that $h_{r-p} \neq 0$ and $h_{r-p+1} = 0$ for the h-vector of $IN(M)$.
(b) Prove part (*ii*). Show that equality holds if and only if $n = r+1$. [Hint: Prop. 7.5.3 is of use.]
(c) Show that for $r \geq 3$:

$$I_1 < I_2 < \ldots < I_{[r/2]} \leq I_{[r/2]+1},$$

where if r is even the last inequality presupposes that M has fewer than $r/2$ isthmuses. The last inequality is strict if r is odd or if $n > r+1$. [Cf. Exercise 7.2.2]

7.5.3. This exercise refers to Proposition 7.5.5.
(a) If M has exactly p connected components, show that $h_{r-p} \neq 0$ and $h_{r-p+1} = 0$, for the h-vector of $BC_\omega(M)$.
(b) Prove part (*ii*). Characterize the case of equality. [Hint: Prop. 7.5.7 is of use.]
(c) Show that for $r \geq 3$:

$$\tilde{w}_1 < \tilde{w}_2 < \ldots < \tilde{w}_{[r/2]} \leq \tilde{w}_{[r/2]+1},$$

where if r is even the last inequality presupposes that M has fewer than $r/2$ connected components. The last inequality is strict if r is odd or if M is connected and $n > r+1$. [Cf. Exercise 7.2.2].
(d) Show that $\tilde{w}_{r-1} > \tilde{w}_r$. [Hint: Here and for the $k = 1$ case in (b) use that $BC_\omega(M)$ is a cone.]
(e) Deduce that the sequence $(\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_r)$ is unimodal for $r \leq 6$.

7.5.4. Prove that the sequence $(I_0, I_1, \ldots, I_r)$ is unimodal for $r \leq 7$.
[Dowling (1980)]

7.5.5. Suppose that a simple matroid M of rank r and size n has exactly q circuits of the minimal cardinality $c = c_M$. Show the following:
(a) $q \leq \binom{n-r+c-1}{c}$,
(b) $I_k \geq \Sigma_{i=0}^{c}\binom{n-r+i-1}{i}\binom{r-i}{k-i} - q\binom{r-c}{r-k}$, for $0 \leq k \leq r$,
(c) $\tilde{w}_k \geq \Sigma_{i=0}^{c-1}\binom{n-r+i-1}{i}\binom{r-i}{k-i} - q\binom{r-c+1}{r-k}$, for $0 \leq k \leq r$.

7.5.6. Prove that $b(M) \geq \frac{n}{r} \cdot \tilde{\mu}(M)$, with equality if and only if M is an r-uniform matroid.
[Björner (1982)]

7.5.7. Show that inequality (7.22) is true for all matroids without loops and isthmuses with only the following exceptions: $C_r \oplus C^*_{n-r}$, for $2 \leq r \leq n-2$, $C_2^2 \oplus C_3^*$, $C_2^2 \oplus C_3$, C_2^3 and C_2^4. Here C_i denotes the i-point circuit and C_2^k is the direct product of k copies of C_2. Show also that in these cases (7.22) fails by at most 2 units.

7.5.8. Let L be a geometric lattice of rank r.
(a) Linearly order the atoms of L and for each $x \in L$ let B_x be the lexicographically first basis of the interval $[\hat{0}, x]$. Show that $\{B_x : x \in L\}$ is a simplicial

complex. [R. Stanley, 1987 (unpublished)]
(b) Show that $(W_0, W_1, \ldots, W_r)$ is the f–vector of a simplicial complex, where $W_k = \text{card } L^k$ are the Whitney numbers of the second kind. [Wegner (1984)]
(c) Show that the complex in (a) is a subcomplex of the broken circuit complex.

7.6.1. Deduce from Proposition 7.6.4 that $(-1)^{r(y)-r(x)}\ \mu(x,y) > 0$ for all $x \leq y$ in a geometric lattice L. [The inequality is due to Rota (1964).]

7.6.2. Let L be a geometric lattice. Show that the order complex $\Delta(\bar{L} - A)$ is shellable for the following subsets $A \subseteq \bar{L}$:
(a) $A = [x, \hat{1}]$, for some $x \in L - \hat{0}$. [Wachs and Walker (1986)]
(b) A is any subset containing no k-element antichain, supposing that every line of L has at least k points. [Baclawski (1982)]
(c) A is the antichain of maximal complements of a fixed element $x \in \bar{L}$.[Björner (1980a)]

7.6.3. Show that the k-tuple $\lambda_\omega(\mathbf{c})$ of (7.28), considered as an unordered set of atoms, contains no broken circuit.

7.6.4. For a geometric lattice L, let $\mathcal{F} = \{\mathcal{R}(\mathbf{c}) : \mathbf{c} \in \mathcal{M}\}$, where $\mathcal{M}$ is the set of maximal chains and $\mathcal{R}$ is the restriction operator (7.29).
(a) Show that $\mathcal{F}$ is a simplicial complex.
(b) Conclude that the h–vector of $\Delta(\bar{L})$ equals the f–vector of $\mathcal{F}$.
(c) Show that $\mathcal{F}$ in general is neither pure nor connected.
[Björner, Frankl and Stanley (1987)]

7.6.5. Let L be a geometric lattice of rank r, and for subsets $J \subseteq \{1, 2, \ldots, r-1\}$ define the *rank-selected* subposet $L^J = \{x \in L : r(x) \in J\} = \bigcup_{j \in J} L^j$. Let $\Delta(L^J)$ denote the order complex of L^J. Prove the following:
(a) $\Delta(L^J)$ is a pure $(|J| - 1)$–dimensional complex.
(b) $\mu(L^J) = \chi(\Delta(L^J)) = (-1)^{|J|-1} \cdot \#\{\mathbf{c} \in \mathcal{M} : \mathcal{D}(\lambda(\mathbf{c})) = J\}$.
[Here $\mu(L^J)$ denotes the Möbius function value $\mu(\hat{0}, \hat{1})$ computed on the poset $L^J \cup \{\hat{0}, \hat{1}\}, \lambda = \lambda_\omega$ is any labeling (7.28) of the set $\mathcal{M}$ of maximal chains in L, and $\mathcal{D}(\lambda_1, \lambda_2, \ldots, \lambda_r) = \{i : \lambda_i > \lambda_{i+1}\}$.]
(c) $\Delta(L^J)$ is shellable.
(d) Determine the homology of $\Delta(L^J)$.
(e)* Describe in matroid-theoretic terms a "natural" basis for the homology group $H_{|J|-1}(\Delta(L^J))$ consisting of fundamental cycles of spherical subcomplexes.
[Part (b) is from Stanley (1974), (c) and (d) from Björner (1980a). See also Stanley (1986).]

7.6.6. Let L be a supersolvable geometric lattice with M–chain $\hat{0} = m_0 < m_1 < \ldots < m_r = \hat{1}$ (for the definition see Exercise 7.4.5). Define an edge-labeling $\lambda_M : Cov(L) \longrightarrow \{1, 2, \ldots, r\}$ by the rule

$$\lambda_M\ (x, y) = \min\{i : m_i \vee x = m_i \vee y\},$$

and extend to a labeling λ_M of unrefinable chains as in (7.28). Show the following:
(a) Lemma 7.6.2 holds for λ_M.
(b) Proposition 7.6.4 holds for λ_M, as well as its generalization in Exercise 7.6.5 (b).
(c) Theorem 7.6.3 holds for λ_M and the natural lexicographic order of labels.
[The definition of λ_M and part (b) are from Stanley (1972), part (c) is from Björner (1980a)]

7.6.7. Let L be a supersolvable geometric lattice as in the preceding exercise. For $x \in L$ define

$$\Lambda(x) = \{i : m_{i-1} \vee x = m_i \vee x\} \subseteq \{1, 2, \ldots, r\},$$

(a generalized Schubert symbol). Show that
(a) $|\Lambda(x)| = r(x)$, for all $x \in L$,
(b) $\{\lambda_M(x, y)\} = \Lambda(y) - \Lambda(x)$, for all $(x, y) \in Cov(L)$.

Now, let P be any partial ordering of $\{1, 2, \ldots r\}$ such that $i < j$ in P implies $i < j$ in $\mathbb{N}$. Define

$$L_P = \{x \in L : \Lambda(x) \text{ is an order ideal in } P\}.$$

Show the following:
(c) Every maximal chain in L_P has length r. Consequently, the order complex $\Delta(\bar{L}_P)$ is pure $(r-2)$–dimensional.[$\bar{L}_P = L_P - \{\hat{0}, \hat{1}\}$.]
(d) The labeling λ_M of maximal chains of L constructed in the preceding exercise restricts to a labeling of the maximal chains of L_P for which (the analogues of) Theorem 7.6.3 and Proposition 7.6.4 hold. In particular, $\Delta(\bar{L}_P)$ is shellable.
(e) If L is Boolean then L_P is a distributive lattice, and every finite distributive lattice arises this way.
(f) If L is the subspace lattice of a 4-dimensional vector space, and P is the ordering of $\{1, 2, 3, 4\}$ whose only comparability relation is $2 < 3$, then L_P is not a lattice.
[A. Björner and R. Stanley, 1985 (unpublished). See also Exercise 49b on p. 164 of Stanley (1986). Part (e) is the fundamental theorem for finite distributive lattices, due to G. Birkhoff (ibid., p. 106). When L is the subspace lattice of a vector space and P is a preordered linear forest, the poset L_P coincides with a quotient of a Tits building of type A, as studied by Wachs (1986). Part (f) answers a question left open by Wachs.]

7.6.8. Let $\Pi = \Pi_{r+1}$ be the lattice of partitions of the set $\{0, 1, \ldots, r\}$ ordered by refinement. A covering relation $x < y$ in Π corresponds to a merging of two distinct blocks B_1 and B_2 of x into one block $B_1 \cup B_2$ of y. Let

$$\lambda(x, y) = max\{minB_1, minB_2\},$$

for all $(x, y) \in Cov(\Pi)$.
Show the following:
(a) Π is a supersolvable lattice.
(b) The edge-labeling λ of Π is a special case of the general construction in Exercise 7.6.6.
(c) There are exactly $r!$ maximal chains in Π with decreasing labels.
(d) $\mu(\Pi) = (-1)^r \cdot r!$
(e) $\mu(\Pi^J) = (-1)^{|J|-1} \sum_\sigma \sigma_1^* \sigma_2^* \ldots \sigma_r^*$,
where Π^J is the rank-selected subposet defined as in Exercise 7.6.5, and the summation is over all permutations $\sigma \in S_r$ such that $\{i : \sigma_i > \sigma_{i+1}\} = J$, and $\sigma_k^* = \sigma_k - \#\{i : i < k \text{ and } \sigma_i < \sigma_k\}$. [E.g., if $\sigma = 42513$, then $\sigma_1^* \ldots \sigma_5^* = 4 \cdot 2 \cdot 3 \cdot 1 \cdot 1 = 24$.]

Now, let P be a partial ordering of $\{1, 2, \ldots, r\}$ such that $i < j$ in P implies $i < j$ in $\mathbb{N}$. Define

$$\Pi_P = \{(B_1, B_2, \ldots, B_k) \in \Pi : \bigcup_{i=1}^{k} (B_i - \{minB_i\}) \text{ is an order ideal in } P\}.$$

Here $minB_i$ denotes the least element of B_i in the natural ordering of $\mathbb{N}$.
(f) Show that the poset Π_P is a special case of the general construction in Exercise 7.6.7.
(g) Deduce that every rank–selected subposet Π_P^J, for $J \subseteq \{1, 2, \ldots, r-1\}$, has shellable order complex.
(h) Give a formula for $\mu(\Pi_P^J)$.
[Stanley (1972), Björner and Stanley (unpublished)]

7.6.9. Show that the following subsets of the partition lattice Π_r, with the induced ordering, have shellable order complexes:
(a) {partitions whose block sizes are $\equiv 0 (\mathrm{mod}\ k)$}, if k divides r,
(b) {partitions whose block sizes are $\equiv 1 (\mathrm{mod}\ k)$}, for any $k \geq 2$,
(c) {non-crossing partitions}, i.e., partitions such that for any blocks B_1 and B_2 the conditions $x_1, x_3 \in B_1, x_2, x_4 \in B_2$ and $x_1 < x_2 < x_3 < x_4$ imply $B_1 = B_2$.
[For (a) and (b) see Calderbank, Hanlon and Robinson (1986), for (a) also Sagan (1986). For (c) see Björner (1980a). In each case the Möbius function has interesting form, see the cited sources.]

7.6.10. Let L be a semimodular lattice of rank r. Say that two maximal chains in L are *adjacent* if they differ in exactly one element. Placing edges between adjacent pairs we get a graph $\mathcal{M}_L$ whose vertex set is the set of maximal chains in L. Let $\partial(\mathbf{c}, \mathbf{d})$ denote the usual graph distance (i.e., length of shortest connecting path) in $\mathcal{M}_L$.
Prove the following:
(a) The graph $\mathcal{M}_L$ is connected.
(b) $\partial(\mathbf{c}, \mathbf{d}) = \#\{c_i \vee d_j : 0 \leq i, j \leq r\} - r - 1$, for any two chains $\mathbf{c} : c_0 < c_1 < \ldots < c_r$ and $\mathbf{d} : d_0 < d_1 < \ldots < d_r$. [Abels (1989)]
(c) diam $(\mathcal{M}_L) = \max_{\mathbf{c},\mathbf{d}} \partial(\mathbf{c}, \mathbf{d}) \leq \binom{r}{2}$. [Björner (1980a)]
(d) If L is geometric, then diam $(\mathcal{M}_L) = \binom{r}{2}$.

7.6.11. Let a *Boolean packing* of a rank r geometric lattice L mean a family of injective and cover–preserving mappings $\varphi_i : B_i \longrightarrow L$ from finite Boolean lattices $B_i, 1 \leq i \leq t$, such that
(*i*) $r(\varphi_i(\hat{0})) + r(\varphi_i(\hat{1})) \geq r$, for all $1 \leq i \leq t$, and
(*ii*) L is the disjoint union of the images $\varphi_i(B_i)$.
(a) Show that every rank 3 geometric lattice has a Boolean packing.
(b)* Is the same true for rank ≥ 4?

7.7.1. The 3–element sets 123, 125, 126, 134, 136, 145, 234, 235, 246, 356, 456 are the facets of a pure 2–dimensional simplicial complex Δ on the vertex set $\{1, \ldots, 6\}$. Show that:
(a) Δ is shellable,
(b) $\Delta - \{123\}$ is not shellable. [Hint: This is a triangulation of a surface. Which one?]
(c) $H_2(\Delta) \cong \mathbf{Z}$, and in the generating 2–cycle (unique up to sign) the facet 123 has coefficient ± 2, while all other facets have coefficient ± 1.

7.7.2.* Find general conditions on a shellable complex Δ (or on a shelling) that guarantee that the basic cycles $\rho_1, \ldots, \rho_p$ of Theorem 7.7.2 are the fundamental cycles of subcomplexes of Δ homeomorphic to spheres, whose union is Δ. [This is the case e.g. if Δ is a pseudomanifold (Danaraj and Klee, 1974), for the lexicographic shelling of $\Delta(\bar{L})$ (Theorem 7.9.3. and Exercise 7.9.2), for the lexicographic shelling of $IN(M)$ (Theorem 7.8.4), and for the shellings of Tits buildings and their type-selected subcomplexes considered in Björner (1984).]

7.8.1. For any matroid M, show that:
(a) $IN(M)$ is acyclic $\Longleftrightarrow$ M has an isthmus $\Longleftrightarrow$ $IN(M)$ is a cone.
(b) $NS(M)$ is acyclic $\Longleftrightarrow$ M has a loop $\Longleftrightarrow$ $NS(M)$ is a cone.
(c) $\overline{BC}_\omega(M)$ is acyclic $\Longleftrightarrow$ M is not connected $\Longleftrightarrow$ $\overline{BC}_\omega(M)$ is a cone.

Here $NS(M)$ is the complex of nonspanning subsets, and ω is an arbitrary ordering of the ground set. [For part (b), cf. Exercise 7.9.5.]

7.8.2. Show that if a cycle $\sigma_{B',\omega'}$, for arbitrary basis B' and ordering ω', is expressed in the basis $\{\sigma_{B,\omega} : i(B) = 0\}$ of Theorem 7.8.4 then all coefficients must equal $-1, 0$ or $+1$.

7.8.3. (a) Show that the mapping $\varphi : B \longrightarrow S - B$ of Proposition 7.8.3 is never injective, if M is a simple matroid. (Equivalently, the sphere $\sum_{B,\omega}$ is never a hyperoctrahedron.)
(b)* Is it true that for any two bases $B, B' \in IN(M)$ there exists some spherical subcomplex $\sum \subseteq IN(M)$ such that $B, B' \in \sum$?

7.8.4. Compute the homology of reduced rooted complexes $\overline{RC}_\pi(L)$. These are defined by the property that $RC_\pi(L)$ is a cone over $\overline{RC}_\pi(L)$, see Exercise 7.4.9.
[Björner and Ziegler (1987)]

7.8.5. (a) For every simplicial complex Δ on a vertex set V such that $V \notin \Delta$, let $\Delta^* = \{A \subseteq V : V - A \notin \Delta\}$. Show that $\Delta^{**} = \Delta$.
(b) Deduce from Alexander duality on the $(n-2)$–dimensional sphere, where $n = |V|$, that $\tilde{H}_i(\Delta) \cong \tilde{H}_{n-3-i}(\Delta^*)$, for all $i \in \mathbf{Z}$. (Here we use reduced homology with coefficients in a *field*.)
(c) Let M be a matroid of cardinality n and rank ≥ 1. If $\Delta = NS(M)$, the complex of nonspanning subsets, then $\Delta^* = IN(M^*)$. Compare the duality $\tilde{H}_i(NS(M)) \cong \tilde{H}_{n-3-i}\,(IN(M^*))$ to the result of Exercise 7.9.5 in view of Theorems 7.8.1 and 7.9.1.
(d) Let M be a matroid of rank ≥ 2. Find a relationship between the homology of the complex of subsets contained in some cocircuit and the complex of subsets not containing any hyperplane.
(e) Generalize parts (c) and (d) to greedoids (Chapter 8).
[For Alexander duality see e.g. Munkres (1984), p. 432. We require field coefficients here only to get a simpler statement avoiding cohomology.]

7.8.6*. Give an explicit combinatorial construction of a basis for the homology of
(a) the reduced broken circuit complex $\overline{BC}_\omega(M)$ of a connected ordered matroid $M(S,\omega)$,
(b) the dual complex G^* of a greedoid G (cf. Section 8.6.3),
(c) the order complex of a geometric semilattice, as defined by Wachs and Walker (1986).

7.8.7. Show that the following conditions are equivalent for a matroid M:
(a) $IN(M)$ is homeomorphic to a sphere,
(b) M is a direct product of circuits,
(c) every independent set of corank one is contained in exactly two bases,
(d) $\tilde{\mu}(M^*) = 1$,
(e) $IN(M)$ has the homology of a sphere.
[For (c) $\Longrightarrow$ (b) see Provan and Billera (1982)]

7.8.8. (a) Give an example of a matroid $M(S)$ and two orderings ω and ω' of S such that $\overline{BC}_\omega(M)$ is homeomorphic to a sphere while $\overline{BC}_{\omega'}(M)$ is not.

(b)* Characterize those ordered matroids $M(S,\omega)$ for which $\overline{BC}_\omega(M)$ is homeomorphic to a sphere.
[It follows from Brylawski (1971) that a necessary condition for (b) is that M is the matroid of a planar graph without a K_4 minor, see also Welsh (1976), p. 237.]

7.9.1. Show that the following conditions are equivalent for a semimodular lattice L:
(a) $\Delta(\bar{L})$ is homeomorphic to a sphere,
(b) L is Boolean,
(c) every interval in L of length two has cardinality four,
(d) $\mu(x,y) = (-1)^{r(y)-r(x)}$, for all $x \leq y$ in L.
If L is known to be geometric then also the following conditions are equivalent to the preceding ones:
(e) every coline is covered by exactly two copoints,
(f) $|\mu(\hat{0},\hat{1})| = 1$,
(g) $\Delta(\bar{L})$ has the homology of a sphere.

7.9.2. Let L be a geometric lattice of rank r. For each basis $B \subseteq L^1$ let $\sum^B$ be the subcomplex of $\Delta(\bar{L})$ generated by the maximal chains $\mathbf{c}_{B,\pi}$, for all $\pi \in S_r$, see (7.43). Show the following:
(a) $\sum^B$ is homeomorphic to the $(r-2)$–sphere.
(b) $\Delta(\bar{L}) = \bigcup \sum^B$, with union over all *nbc*–bases B.
(c) The fundamental cycle of $\sum^B$ is equal to the elementary cycle ρ_B (up to sign, see (7.44)).
(d) If ρ_B is expressed in the basis $\{\rho_A : A \in \mathbf{nbc}\}$ of Theorem 7.9.3 then all coefficients must equal $-1, 0$, or $+1$.

7.9.3. Let $\mathcal{B}$ be a neat base–family in a geometric lattice L of rank r, as defined in Exercise 7.4.8. Show that the elementary cycles $\{\rho_A : A \in \mathcal{B}\}$ form a basis for $H_{r-2}(L)$.
[Björner (1982)]

7.9.4. The set of bases $\mathcal{B}_L$ of a geometric lattice L has the structure of a simple matroid induced by linear independence of the elementary cycles $\rho_B, B \in \mathcal{B}_L$, in the free Abelian group $H_{r-2}(\bar{L})$.
Show the following:
(a) For every subset $\mathcal{F} \subseteq \mathcal{B}_L$, if $\{\rho_B : B \in \mathcal{F}\}$ is a basis of $H_{r-2}(\bar{L})$ then $\mathcal{F}$ is a (matroid) basis of $\mathcal{B}_L$.
(b) If L is the lattice of the 3–uniform matroid of size 6, then the matroid $\mathcal{B}_L$ is not regular.
(c) Deduce from (b) that the converse to (a) is in general false.
(d) $\mathcal{B}_L$ is 2–partitionable. [A matroid $M(S)$ is *2–partitionable* if for every $x \in S$ there is a partition $S - x = S_1 \cup S_2, S_1 \cap S_2 = \emptyset$, such that $x \notin \bar{S}_1$ and $x \notin \bar{S}_2$.]
[Björner (1982), Lindström (1981). See also Section 6.5 of White (1987).]

7.9.5. Let $M = M(S)$ be a simple matroid and L the corresponding geometric lattice of flats. The nonspanning subsets of S form a simplicial complex $NS(M)$.
(a) Show that $NS(M)$ and $\Delta(\bar{L})$ have isomorphic homology groups in all dimensions. [Folkman (1966)]
(b) Deduce Theorem 7.9.1 from (a). [Folkman (1966)]
(c) Show that $NS(M)$ and $\Delta(\bar{L})$ are of the same homotopy type. [Lakser (1971)]

7.9.6. Let M be an infinite matroid of rank r and let L be the corresponding geometric lattice.
(a) Show that $H_i\ (IN(M)) = 0$ for $i < r-1$ and $H_i(\Delta(\bar{L})) = 0$ for $i < r-2$.

(b) Show that $H_{r-1}(IN(M))$ and $H_{r-2}(\Delta(\bar{L}))$ are free Abelian groups and determine their ranks.
(c) Define shellability for infinite finite-dimensional simplicial complexes in a reasonable way. Prove the basic properties of this concept.
(d) Show that $IN(M)$ and $\Delta(\bar{L})$ are shellable.
(e) Define $BC_\omega(M)$ and $\overline{BC}_\omega(M)$ for a well-ordering ω of the ground set. Develop the basic theory of infinite broken circuit complexes.
(f) Show that $\overline{BC}_\omega(M)$ is shellable and compute its homology.
[Björner (1982, 1984), Wachs and Walker (1986)]

7.9.7. Let L be a geometric lattice. For $1 \leq k \leq r$ let $D_k(L)$ be the Abelian group freely generated by all k-chains $x_1 < x_2 < \ldots < x_k$ in $L - \hat{0}$. Put $D_0(L) = \mathbf{Z}$, and $D_k(L) = 0$ for all $k < 0$ and all $k > r$. For $2 \leq k \leq r$ define group homomorphisms $d_k^W : D_k(L) \longrightarrow D_{k-1}(L)$ on basis elements by

$$d_k^W\ (x_1 < x_2 < \ldots < x_k) = \sum_{i=1}^{k-1} (-1)^i\ (x_1 < \ldots < \hat{x}_i < \ldots < x_k),$$

and extend linearly to all of $D_k\ (L)$. Let $d_k^W = 0$ for all other k. Then $d_{k-1}^W \circ d_k^W = 0$, for all $k \in \mathbf{Z}$ (check this). The homology of this algebraic chain complex,

$$H_k^W(L) = Ker\ d_k^W / Im\ d_{k+1}^W,$$

is the *Whitney homology* of L.
(a) Show that

$$H_k^W\ (L) \cong \begin{cases} \oplus_{x \in L - \hat{0}}\ H_{k-2}(\hat{0}, x), & \text{if } k \neq 0 \\ \mathbf{Z}, & \text{if } k = 0. \end{cases}$$

(b) Conclude that the definition (7.45) of Whitney homology is equivalent to the one given here.
[Baclawski (1975), Björner (1982), Orlik and Solomon (1980)]

7.10.1. Let L be a geometric lattice and e an atom which is not an isthmus. Let $L - e$ and L/e denote the geometric lattices of the deletion and contraction by e, respectively. Show that there exist linear maps giving short exact sequences of algebras:
(a) $0 \longrightarrow \mathcal{A}(L - e) \longrightarrow \mathcal{A}(L) \longrightarrow \mathcal{A}(L/e) \longrightarrow 0$,
(b) $0 \longrightarrow H^W(L - e) \longrightarrow H^W(L) \longrightarrow H^W(L/e) \longrightarrow 0$.
[Orlik, Solomon and Terao (1984), Jambu and Terao (1989)]

7.10.2. Let $RC_\pi(L)$ be a rooted complex in a geometric lattice L, as defined in Exercise 7.4.9. Show that $\{\bar{e}_A : A \in RC_\pi(L)\}$ is a linear basis for the Orlik-Solomon algebra $\mathcal{A}(L)$.
[Björner and Ziegler (1987). *Hint*: An alternative to a direct argument is to use Theorem 7.10.2 (*i*) together with Exercises 7.4.9 (f) and 7.9.3.]

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