MAT 320 **Finite Topological Spaces**
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Definition Let X be a finite set. A *topology on* X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$ of X satisfying these three conditions:

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(ii) if $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cup V \in \mathcal{T}$.

(iii) if $U \in \mathcal{T}$ and $V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

The ordered pair $(X, \mathcal{T})$ is called a *finite topological space*. The set X is called the *underlying set* of the topological space $(X, \mathcal{T})$.

The indiscrete topology If X is any finite set, then $\mathcal{T} = \{\emptyset, X\}$ is a topology on X - conditions (ii) and (iii) above are satisfied vacuously. This is called the *indiscrete topology* on X.

The discrete topology If X is any finite set, then $\mathcal{T} = \mathcal{P}(X)$ is a topology on X, called the *discrete topology* on X.

Example Let $X = \{0, 1\}$. Then there are four different topologies on X; they are listed below. The first two on the list are the indiscrete and discrete topologies, respectively.

$\mathcal{T}_1 = \{\emptyset, \{0, 1\}\}$.

$\mathcal{T}_2 = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

$\mathcal{T}_3 = \{\emptyset, \{0\}, \{0, 1\}\}$.

$\mathcal{T}_4 = \{\emptyset, \{1\}, \{0, 1\}\}$.

The pair $(X, \mathcal{T}_3)$ is called the *Sierpinski space*.

Open and closed sets Suppose $(X, \mathcal{T})$ is a finite topological space. A subset U of X is called an *open set* if U is an element of $\mathcal{T}$. A subset K of X is called a *closed set* if its complement $K^c = X - K$ is an element of $\mathcal{T}$.