This homework concerns the easy part of Taylor’s series.

1) Find the Taylor’s series by *trickery* based on the geometric series.
   a) \( f(x) = \frac{1}{1-x} \)
   b) \( f(x) = \frac{1}{(7-x)^2} \)
   c) \( f(x) = \frac{x}{7-x} \)

2) Find the Taylor’s series by *trickery* based on your knowledge of the series for \( e^x, \sin x \) and \( \cos x \).
   a) \( f(x) = \cos(2x) \)
   b) \( f(x) = \frac{\sin(x)}{x} \)
   c) \( f(x) = \cos(\sqrt{x}) \)
   d) \( f(x) = \sin(\sqrt{x}). \) Factor out \( \sqrt{x}. \)
   e) \( f(x) = \sqrt{x}\sin(\sqrt{x}). \)
   f) \( f(x) = 2xe^{x^2} \)

3) Use the Taylor’s series from problem 2 to find Taylor’s series for the integrals and use them to compute the definite integrals. Do the definite integrals using three terms and 4 terms of the series.
   a) \( \int \frac{\sin(x)}{x} \, dx \)
   b) \( \int_0^\pi \frac{\sin(x)}{x} \, dx \) Ans: \( \ldots \frac{2}{3!} \ldots \)
   c) \( \int \cos(\sqrt{x}) \, dx \) Ans: \( \ldots \frac{2}{3!} \ldots \)
   a) \( \int_0^\pi \cos(\sqrt{x}) \, dx \) Ans: 3 terms .45705; 4 terms .45703

4) Use the Taylor series formula
   \[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \]
   to find the Taylor’s series of the following functions. Find at least the first 5 terms and see if you can guess the general form of the terms. Some of these are very important in applications.
   a) \( \ln(1-x) \quad a = 0 \)
   b) \( \sqrt{1+x} \quad a = 0 \)
   c) \( \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} \quad a = 0 \)
   d) \( \frac{1}{\sqrt{1-x^2}} \quad a = 0 \) Substitute \(-x^2\) for \( x \) in previous problem
   e) \( \ln x \quad a = 1 \)
   f) \( \sin x \quad a = \frac{\pi}{4} \)
5) Now the remainder term problems. Remember the error estimate in a Taylor’s series is given by

\[ |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \]

where \(|f^{(n)}(x)| \leq M\) on the interval of interest. Using this, estimate the maximum possible error in using the first the Taylor polynomial \(T_n(x)\) for the function on the given interval. The error may be a lot less than the error estimate indicates.

a) \(f(x) = e^x, a = 0\), on the interval \(-\frac{3}{2} \leq x \leq \frac{3}{2}\)  \ Ans: \( M = 4.48169 \) and \( |\text{error}| \leq 0.07090 \)

b) \(f(x) = \ln(1-x), a = 0\), on the interval \(-\frac{1}{2} \leq x \leq \frac{1}{2}\)  \ Ans(?) : \( M = 7680 \) and \( |\text{error}| \leq 0.16666 \)

c) \(f(x) = \cos(x), a = 0\), on the interval \(-\frac{3}{2} \leq x \leq \frac{3}{2}\)  Students often muck this one up because there are zero terms in the Taylor’s series and \(T_5(x)\) is the same as \(T_4(x)\). This has NO effect on the error term; you still estimate \(|R_5(x)|\) as usual.  \ Ans: \( M = 1 \) and \( |\text{error}| \leq 0.01582 \). This is quite close to the actual error which is .01520

d) \(f(x) = \sin(x), a = 0\), on the interval \(-\frac{3}{2} \leq x \leq \frac{3}{2}\). Students often muck this one up for different reasons. Here, \(T_5(x)\) is the same as \(T_6(x)\) since the \(k = 6\) term is 0. Hence the error term as calculated from \(|R_5(x)|\) would be much too large; a better estimate for the error comes from calculating \(|R_6(x)|\) so do this.  \ Ans: \( M = 1 \) and \( |\text{error}| \leq 0.00339 \)

There is something slightly funny here; \(\sin\left(\frac{3}{2}\right) = .99749\) but the \(T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}\) gives a value of 1.00078, which is embarrassing for a value of \(\sin\). Nevertheless, the error \(|T_5\left(\frac{3}{2}\right) - \sin\left(\frac{3}{2}\right)| = 1.00078 - .99749 = .00329\) is less than our estimate .00339 so everything is in order.

6) And finally the ultimate: How many terms of the series of \(e^x\) would you have to take so that on the interval \([-1.2,1.2]\) the error would be less than .0005? You do this by finding the first \(n\) for which \(|R_n(x)| \leq .0005\) on the given interval. This is not as hard as it looks. The answer is a number whose square is the cube of half of itself.