1. Pictured below is a triangulation of the projective plane, with 6 vertices, 15 edges, and 10 2-simplices. Use this triangulation to calculate (on a separate page) the homology, i.e., determine the betti numbers,

\[ \begin{array}{c}
\mathbb{C}_2 \xrightarrow{\partial_2} \mathbb{C}_1 \xrightarrow{\partial_1} \mathbb{C}_0 = \mathbb{R}^{10} \xrightarrow{\partial_2} \mathbb{R}^{15} \xrightarrow{\partial_1} \mathbb{R}^6 \\
\end{array} \]

(a) with real coefficients, and

(b) with coefficients in the field \( \mathbb{Z}_2 = \{0, 1\} \). Note, in \( \mathbb{Z}_2 \), \(-1\) is equal to 1.

(a) \[ \dim H^2 = \dim(\ker \partial_2) = 10 - \text{rank}(\partial_2) = 10 - 10 = 0 \]

\[ \dim H^1 = \dim(\ker \partial_1) - \dim(\text{im} \partial_2) = (15 - \text{rk}(\partial_1)) - \text{rk}(\partial_2) = (15 - 5) - 10 = 0 \]

\[ \dim H_0 = 6 - \text{dim}(\text{im} \partial_1) = 6 - \text{rk}(\partial_1) = 6 - 5 = 1 \]

(b) \( H_0(\mathbb{P}^2, \mathbb{Z}_2) = \mathbb{Z}_2 \) for \( i=0, 1, 2 \), \( H_i(\mathbb{P}^2, \mathbb{Z}_2) = 0 \) otherwise.

2. Poincaré proved a duality theorem for betti numbers of compact manifolds without boundary: if \( \beta_i \) denotes the \( i \)th betti number of \( M \), then, if \( M \) is a compact manifold of dimension \( n \), \( \beta_i = \beta_{n-i} \).

(This is called "Poincaré duality.") Use this result to show that the Euler-Poincaré characteristic of any three-dimensional compact manifold without boundary is zero.

\[ \chi(M^3) = \beta_0 - \beta_1 + \beta_2 - \beta_3 \]

by Thom from class, and \( \beta_0 = \beta_3 \) and \( \beta_1 = \beta_2 \) by Poincaré duality, so \[ \chi(M^3) = \beta_0 - \beta_1 + \beta_1 - \beta_0 = 0. \]
3. Do Exercise 6.1.4 from the text.

Let \( \alpha, \beta : [0,1] \rightarrow S^2 \) with \( \alpha(0) = \beta(0) = x_0 \) and \( \alpha(1) = \beta(1) = x_1 \). Assume \( \alpha(s) \neq -\beta(s) \) for any \( s \in [0,1] \). Define

\[
H(s, t) = \frac{(1-t)\alpha(s) + t\beta(s)}{\| (1-t)\alpha(s) + t\beta(s) \|}
\]

Note \( (1-t)\alpha(s) + t\beta(s) \neq 0 \) for any \( (s,t) \in [0,1] \times [0,1] \) since \( \alpha(s) \neq -\beta(s) \). (Else \( \| (1-t)\alpha(s) \| = \| -t\beta(s) \| \), so \( (1-t)\|\alpha(s)\| = t\|\beta(s)\| \), \( (1-t) = t \), \( t = \frac{1}{2} \) and then \( \alpha(s) = -\beta(s) \)). Thus \( H : [0,1] \times [0,1] \rightarrow S^2 \) is well-defined. It is easy to show \( H \) is a path-homotopy from \( \alpha \) to \( \beta \).

\[\begin{array}{cccccccc}
12 & 124 & 125 & 134 & 136 & 156 & 235 & 236 & 246 & 345 & 456
\end{array}\]

\( \partial_2 = \)

\( \begin{bmatrix}
12 & 124 & 125 & 134 & 136 & 156 & 235 & 236 & 246 & 345 & 456
\end{bmatrix} \)

Using Mathematica
MatrixRank
(And Modulus \rightarrow 2)

\( \text{rank}(\partial_2) = 10 \quad \text{rank}_{\mathbb{F}_2}(\partial_2) = 9 \)

\( \partial_1 = \)

\( \begin{bmatrix}
\end{bmatrix} \)

\( \text{rank}(\partial_1) = 5 \quad ; \quad \text{rank}_{\mathbb{F}_2}(\partial_1) = 5 \)