Definition Let $X$ be and $Y$ be topological spaces, and let $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$. The product topology on $X \times Y$ is defined as follows: a subset $W \subseteq X \times Y$ is open if and only if, for every $(x,y) \in W$ there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x,y) \in U \times V \subseteq W$.

1. Show that $X \times Y$ is Hausdorff if both $X$ and $Y$ are Hausdorff.

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $(x_1, y_1) \neq (x_2, y_2)$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Suppose $x_1 \neq x_2$. Since $X$ is Hausdorff, there exist open sets $U_1, U_2$ in $X$ with $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$. Then $U_1 \times Y$ and $U_2 \times Y$ are open in $X \times Y$, $(x_1, y_1) \in U_1 \times Y$, $(x_2, y_2) \in U_2 \times Y$, and $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \emptyset$.

Similarly, if $y_1 \neq y_2$ then there are disjoint open sets $X \times V_1$ and $X \times V_2$ in $X \times Y$ containing $(x_1, y_1)$ and $(x_2, y_2)$, respectively (using the hypothesis that $Y$ is Hausdorff).

$\Rightarrow$ Suppose $X$ is Hausdorff.

Let $(x, y) \in X \times X - \Delta$. Then $x, y \in X$ and $x \neq y$, so there exist open nbhds $U, V$ of $x$ and $y$, respectively, in $X$, with $U \cap V = \emptyset$. Then $U \times V$ is an open nbhd of $(x, y)$ in $X \times X$. Claim $U \times V \subseteq X \times X - \Delta$. If not, then $(U \times V) \cap \Delta \neq \emptyset$, so there exist $(p, p) \in \Delta$ with $(p, p) \in U \times V$, equivalently, $p \in U \cap V$. Since $U \cap V = \emptyset$ this cannot be. Thus $U \times V \subseteq X \times X - \Delta$. This shows $X \times X - \Delta$ is open, so $\Delta$ is closed. $\Leftarrow$

Let $x, y \in X$ with $x \neq y$. Then $(x, y) \in X \times X - \Delta$. Since $\Delta$ is closed by hypothesis, $X \times X - \Delta$ is open, and there are open subsets $U$ and $V$ in $X$ such that $(x, y) \in U \times V$ and $U \times V \subseteq X \times X - \Delta$. Then $x \in U$ and $y \in V$, and, as above, $U \cap V = \emptyset$. Thus $X$ is Hausdorff.
3. Let \( f : X \to Y \) be a continuous function, with \( Y \) Hausdorff. The graph of \( f \) is the subset

\[ \Gamma(f) = \{(x,y) \in X \times Y \mid y = f(x)\} \]

of \( X \times Y \). Show that \( \Gamma(f) \) is a closed subset of \( X \times Y \).

We show \( X \times Y - \Gamma(f) \) is open in \( X \times Y \). Let \( (x,y) \in X \times Y - \Gamma(f) \). Then \( y \neq f(x) \). Since \( Y \) is Hausdorff, there are open sets \( U \) and \( V \) in \( Y \) such that \( f(x) \in U \), \( y \in V \), and \( U \cap V = \emptyset \). Then \( f^{-1}(U) \) is an open nbhd of \( x \) in \( X \), since \( f \) is continuous and \( f(x) \in U \). Then \( (x,y) \in f^{-1}(U) \times V \). Claim \( f^{-1}(U) \times V \subseteq X \times Y - \Gamma(f) \). Indeed, if \( (u,v) \in f^{-1}(U) \times V \) then \( f(u) \in U \) and \( v \in V \), so \( f(u) \neq V \), since \( U \cap V = \emptyset \), and so \( (u,v) \notin \Gamma(f) \). This shows \( X \times Y - \Gamma(f) \) is open, hence \( \Gamma(f) \) is closed.

4. Suppose \( f : X \to X \) is continuous, and \( X \) is Hausdorff. Show that the set of fixed points of \( f \) is closed in \( X \).

A fixed point of \( f : X \to X \) is a point \( x \in X \) satisfying \( f(x) = x \).

Let \( F = \{x \in X \mid f(x) = x^2\} \). Let \( x \in X - F \).

Then \( f(x) \neq x \). Since \( X \) is Hausdorff, there are open sets \( U \) and \( V \) in \( X \) with \( x \in U \), \( f(x) \in V \), and \( U \cap V = \emptyset \). Since \( f \) is continuous, \( f^{-1}(V) \) is open in \( X \), and \( x \in f^{-1}(V) \) since \( f(x) \in V \). Then \( x \in U \cap f^{-1}(V) \) and \( U \cap f^{-1}(V) \) is open in \( X \). Claim \( U \cap f^{-1}(V) \subseteq X - F \).

Indeed, if \( y \in U \cap f^{-1}(V) \), then \( y \in U \) and \( f(y) \in V \), so \( y \neq f(y) \), since \( U \cap V = \emptyset \). Thus \( y \in X - F \). This shows \( X - F \) is open, so \( F \) is closed.