1. (20) (a) Show that every integer \( n \) less than or equal to 2 occurs as the Euler-Poincaré characteristic of some compact surface without boundary.

The standard cell structure on \( gT^2 \) has 1 vertex, \( 2g \) 1-cells, and 1 2-cell, so \( \chi(gT^2) = 1 - 2g + 1 = 2 - 2g \). Then, if \( N \) is an even integer, \( N < 2 \), then \( \frac{1}{2}(2-N) = g \) is positive, and \( \chi(gT^2) = 2 - 2g = 2 - (2-N) = N \). The standard cell structure on \( gP^2 \) has 1 vertex, \( g \) 1-cells, and 1 2-cell, so \( \chi(gP^2) = 1 - g + 1 = 2 - g \). Then, if \( N < 2 \), \( g = 2-N \) is positive and \( \chi(gP^2) = 2 - (2-N) = N \). Since

(b) Up to homeomorphism, how many different compact surfaces \( S \) are there with given Euler-Poincaré characteristic \( \chi(S) \), in case

(i) \( \chi(S) \) is even.

If \( \chi(S) = 2 \) there is one.

If \( \chi(S) = N < 2 \) is even, there are two: \( S = gP^2 \) with \( g = 2 - \chi(S) \), and \( S = gT^2 \) with \( g = \frac{1}{2}(2 - \chi(S)) \). (See part (a).)

(ii) \( \chi(S) \) is odd.

There is one: \( S = gP^2 \) with \( g = 2 - \chi(S) \).

(c) Make a list of all compact surfaces with non-negative Euler-Poincaré characteristic.

\( \chi(S^2) = 2 \)
\( \chi(P^2) = 1 \)
\( \chi(T^2) = 0 \)
\( \chi(P^2 \# P^2) = 0 \)

\( \leftarrow \) Klein bottle.

That is all.
2. (20) Each expression below is meant to identify the identification pattern for a planar identification diagram for a topological space. In each case

(i) determine whether the quotient space is a topological surface (without boundary);

(ii) if the quotient space is a surface, determine whether the that surface is orientable or not.

(iii) compute the Euler-Poincaré characteristic of the quotient space (whether or not the quotient space is a surface).

(iv) if the quotient space is a surface, identify it according to the classification theorem for surfaces (i.e., as a connected sum of $T^2$'s and/or $P^2$'s. Hint: Use (iii)

(a) $abca^{-1}b^{-1}a^{-1}b$

There are $3$ 1-cells and $1$ 2-cell; one also finds that there are $2$ vertices, so the Euler-Poincaré characteristic is $2 - 3 + 1 = 0$

(b) $abba^{-1}c^{-1}$

The quotient is a surface, since edges are identified in pairs. There is one vertex in the quotient which also has a neighborhood $\not\cong \mathbb{R}^2$. (See also Prop. 2.6.2)

(c) $abed^{-1}a^{-1}b^{-1}$

The surface is not orientable since $b$ is traversed twice with consistent orientation so the surface contains a Möbius band.

The Euler-Poincaré characteristic is $1 - 3 + 1 = -1$ so the surface is $P^2 \# P^2 \# P^2$ or $T^2 \# T^2$. 

The quotient is a surface because edges are identified in pairs. It is orientable because each pair are oppositely oriented.

There is $1$ vertex, $4$ edges, and $1$ 2-cell; $\chi = 1 - 4 + 1 = -2$ so the surface is $T^2 \# T^2$. 

Not a surface; a neighborhood of $x$ in the quotient looks like $\not\cong \mathbb{R}^2$
3. (15) (a) Suppose $S'$ is a compact surface with boundary, obtained from a compact surface $S$ by removing $k$ open disks (with disjoint closures). Express $\chi(S')$ in terms of $\chi(S)$ and $k$.

Choose a triangulation of $S$ having $k$ disjoint 2-simplices and remove their interiors to form $S'$. Then removing those 2-simplices yields a triangulation of $S'$. Then

$$\chi(S) = \chi(S') + k,$$
so $$\chi(S') = \chi(S) - k.$$

*Note: this can be accomplished by repeatedly subdividing.*

(b) Make a list of all compact surfaces with boundary having non-negative Euler-Poincaré characteristic.

$k \geq 1$ and $\chi(S') \geq 1$, so

$$1 \leq \chi(S') + k = \chi(S) \leq 2 \text{ (using problem 1), so there are these possibilities:}$$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$S'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{P}^2$</td>
<td>$\mathbb{P}^2$, $\mathbb{P}^2 - \mathbb{D}^2$</td>
</tr>
<tr>
<td>$S^2$</td>
<td>$S^2$</td>
</tr>
<tr>
<td>$S^2 \times S^1$</td>
<td>$S^2 \times S^1$</td>
</tr>
<tr>
<td>$D^2$</td>
<td>$D^2$</td>
</tr>
</tbody>
</table>

(c) Identify the surfaces pictured below (from the Math/Stat Department website http://www.cefne.nau.edu/Academic/Math/researchInterests/) by calculating the Euler-Poincaré characteristic and counting the number of boundary components. Construct a cell decomposition by inserting vertices along the boundary components and 1-cells in the surface separating the twists or crossings. These surfaces are orientable - if you look at the colored pictures on the web page you'll see that the surfaces have two sides, one green and one blue.

Note: All surfaces are connected and orientable.

Surface # 1: $\#$ boundary components = 1; $f_0 = 12$, $f_1 = 12 + 6 = 18$,

$$f_2 = 5,$$ $\chi(S') = 12 - 18 + 5 = -1$

so $\chi(S) = -1 + 1 = 0$. $S$ is orientable, so $S = \mathbb{P}^2$ and $S' = \mathbb{P}^2 - \mathbb{D}^2$.

Surface # 3: $\#$ boundary comps = 3: $f_0 = 16$, $f_1 = 16 + 8 = 24$,

$$f_2 = 7,$$ $\chi(S') = 16 - 24 + 7 = -1$

so $\chi(S) = -1 + 3 = 2$. $S = S^2$, $S' = S^2 - 3$ disks.

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1. It can be shown that every compact surface with boundary can be obtained this way.
4.(5) Let $V$ be the set of nonempty subsets of $\{1,2,3\}$. Let $K$ be the abstract simplicial complex with vertex set $V$ whose simplices are the subsets of $V$ which are linearly ordered, that is, for every $A, B \in \sigma$, $A \subseteq B$ or $B \subseteq A$. Show that $K$ determines a triangulation of the standard simplex $\Delta^2$.

5.(10) Let $K$ be the abstract simplicial complex $K = \{1, 2, 3, 4, 12, 13, 23, 24, 34, 123\}^2$. Compute the simplicial homology of $K$ (with real coefficients).

\[
\begin{array}{c}
0 \rightarrow \mathbb{R}^2 \xrightarrow{\partial_2} \mathbb{R}^3 \xrightarrow{\partial_1} \mathbb{R}^4 \rightarrow 0
\end{array}
\]

\[
\partial_2 = \begin{bmatrix}
123 \\
1 & 1 & -1 \\
3 & 1 & 2 \\
2 & 4 & 1 \\
3 & 4 & 0
\end{bmatrix}
\]

\[
\text{rank } \partial_2 = 1, \quad \text{nullity } \partial_2 = 1 - 1 = 0
\]

\[
\Rightarrow H_2(K) = \ker(\partial_2) = 0
\]

\[
\partial_1 = \begin{bmatrix}
12 & 13 & 23 & 24 & 34 \\
-1 & -1 & 0 & 0 & 0 \\
2 & 1 & 0 & -1 & -1 \\
3 & 0 & -1 & 0 & 0 \\
4 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\text{rank } \partial_1 = 3, \quad \text{nullity } \partial_1 = 5 - 3 = 2
\]

\[
H_1(K) = \ker(\partial_1) / \im \partial_2 \cong \mathbb{R}^2 / \mathbb{R} \cong \mathbb{R}^1
\]

and

\[
H_2(K) = \mathbb{R}^4 / \im \partial_1 \cong \mathbb{R}^4 / \mathbb{R}^3 \cong \mathbb{R}
\]

\[\text{We are using the standard shorthand, denoting sets of numbers by listing their elements without commas or set braces.}\]

\[
\text{so } H_c(K) = \begin{cases} 
\mathbb{R} & \text{if } c = 0, \\
0 & \text{otherwise}
\end{cases}
\]
Surface # 2:  \# boundary components = 3
\[ f_0 = 20, \quad f_1 = 20 + 10 = 30, \quad f_2 = 9 \]
\[ \chi(S') = 20 - 30 + 9 = -1 \]
\[ \chi(S) = -1 + 3 = 2, \quad \Rightarrow S = S^2 \]
\[ S' = S^2 - 3 \text{ disks} \]