1. (16) (a) Suppose $X$ is a $T_1$ space\(^1\) and $A \subseteq X$. Show that $(A')' \subseteq A'$.

Note: There is a hint on the web page.

**Lemma.** If $U \subseteq X$ is open and $K \subseteq X$ is closed, then $U - K$ is open. Proof: \(X - K\) is open and $U - K = U \cap (X - K)$. Thus $U - K$ is open.

Now let $x \in (A')'$ and let $U$ be an open nbhd of $x$. Then $U \cap A' \neq \emptyset$. Let $y \in U \cap A'$. If $x = y$ let $V = U$; if $x \neq y$ let $V = U - \{x\}$. Then $V$ is an open nbhd of $y$ and $V \subseteq U$. Since $y \in A'$ we conclude $A \cap (V - \{x\}) \neq \emptyset$. Since $x \not\in A \cap (U - \{y\})$, we conclude that $A \cap (U - \{y, x\}) \neq \emptyset$, hence $x \in A'$. \(\square\)

(b) Show that any finite $T_1$ space is discrete.

Since any finite union of closed sets is closed, any finite subset of a $T_1$ space is closed. So, if $X$ is a finite $T_1$ space, every subset is open, hence every subset is open, hence $X$ is discrete.

(c) Let $X = \{0, 1, 2\}$ with the topology $T = \{\emptyset, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}, X\}$, and let $A = \{1\}$. Show that $(A')' \neq A'$.

\[A' = \{0, 2\}\] since every open nbhd of each of 0 and 2 meets $A$ in a point different of 0 and 2, but not so for 1.

\[(A')' = 0, 2 \neq \emptyset \neq A'.\]

(d) Prove: if $X$ is a $T_1$ space, $A \subseteq X$, and $x \in A'$, then every open neighborhood of $x$ contains infinitely many points of $A$.

Note: There is a hint on the web page.

Suppose $X$ is $T_1$, $A \subseteq X$, $x \in A'$, and $U$ is an open nbhd of $x$. Suppose $U \cap A$ is finite. Then $(U \cap A)' = \{x\}$ is finite, hence closed. Then $V = U - (U \cap A)'$ is open, and $x \in V$, but $A \cap (V - \{x\}) = A \cap (U - (U \cap A)) - \{x\} = \emptyset$, contradicting the assumption that $x \in A'$.

Thus $U \cap A$ is infinite.

\(^1\)That is, $\{x\}$ is closed for every $x \in X$. 
2.(12) Let \( \mathbb{R} \) denote the set of reals with the standard topology, and \( \mathbb{R}_c \) the set of reals with the half-open interval topology.\(^2\)

(a) Let \( f \) be the identity function, \( f(x) = x \) for all real numbers \( x \). Determine whether \( f: \mathbb{R} \to \mathbb{R}_c \) and/or \( f: \mathbb{R}_c \to \mathbb{R} \) are continuous, and prove your answer. 

Let \( -\infty < a < b < \infty \). Then \( (a, b) = \bigcup_{n=1}^{\infty} (a + \frac{1}{n}, b) \), hence \( (a, b) \) is open in \( \mathbb{R}_c \). It follows that any open set in \( \mathbb{R} \) is open in \( \mathbb{R}_c \). Thus \( f: \mathbb{R}_c \to \mathbb{R} \) is continuous. 

On the other hand, \( [a, b) \) is not open in \( \mathbb{R}_c \), so \( f: \mathbb{R} \to \mathbb{R}_c \) is not continuous.

(b) Let \( X = \mathbb{R}_c \times \mathbb{R} \), with the product topology. Let \( L \) be a straight line in \( X \), with the subspace topology. Determine conditions under which \( L \) is homeomorphic to \( \mathbb{R} \), or to \( \mathbb{R}_c \), or neither.

A basic open set in \( \mathbb{R}_c \times \mathbb{R} \) has the form \( (a, b) \times (c, d) \). If \( L \) is not vertical, then \( L \) is homeomorphic to \( \mathbb{R} \), since \( L \cap (a, b) \times (c, d) \) can be a half-open interval (but not a single point). If \( L \) is vertical, then \( L \) is homeomorphic to \( \mathbb{R} \).

(c) Let \( x, y \in \mathbb{R}_c \), with \( x \neq y \). Prove that there are open subsets \( U \) and \( V \) of \( \mathbb{R}_c \) with \( x \in U, y \in V, U \cap V = \emptyset, \) and \( U \cup V = \mathbb{R}_c \). (A space with this property is said to be totally disconnected.)

Assume without loss of generality that \( x < y \). From part (a), \( (a, y) \) is open for all \( a < y \), so

\[
U = (-\infty, y) = \bigcup_{a < y} (a, y) \text{ is open, and it contains } x.
\]

Similarly, \( V = [y, \infty) = \bigcup_{b > y} [y, b) \) is open, and it contains \( x \).

\[
U \cap V = (-\infty, y) \cap [y, \infty) = \emptyset, \text{ and } U \cup V = (-\infty, y) \cup [y, \infty) = \mathbb{R}_c.
\]

\(^2\)So \( \mathbb{R}_c \) has basis consisting of the half-open intervals \([a, b)\) with \(-\infty < a < b < \infty\).
3. (12) (a) Let $X$ and $Y$ be nonempty topological spaces. Suppose $X \times Y$ is Hausdorff. Prove that $X$ is Hausdorff.

Let $x, x' \in X$ with $x \neq x'$. Choose $y \in Y$. (Since $Y \neq \emptyset$ we can do that.) Then $(x, y) \neq (x', y)$. Since $X \times Y$ is Hausdorff, there are disjoint open nbhds $W$ and $W'$ of $(x, y)$ and $(x', y)$, respectively. Then $\exists$ open sets $U, U'$ in $X$ and $V, V'$ in $Y$ with $(x, y) \in U \times V \subseteq W$ and $(x', y) \in U' \times V' \subseteq W'$.

(b) Let $p : X \times Y \to X$ be the canonical projection, $p(x, y) = x$. Show that $p$ is an open map. Then $x \in U$, $x' \in U'$, and $U \cap U' = \emptyset$. Thus $X$ is Hausdorff.

(c) Find an example to show that $p : X \times Y \to X$ need not be a closed map, that is, it need not map closed sets to closed sets.

Note: You may find it convenient to use the results of HW #2.3

Let $X = Y = \mathbb{R}$. Let $K = \{ (x, y) \in \mathbb{R}^2 \mid x = \tan^{-1}(y) \}$. Then $K$ is closed in $\mathbb{R}^2$: $K$ is the image under the homeomorphism $(x, y) \mapsto (y, x)$ of the graph of the continuous function $f(x) = \tan^{-1}(x)$, which is closed by HW 2.3. But $p(K) = (-\frac{\pi}{2}, \frac{\pi}{2})$, which is not closed in $\mathbb{R}$.

4. (5) Let $f : X \to Y$ be a continuous function. Let $\Gamma(f) \subseteq X \times Y$ be the graph of $f$, defined by $\Gamma(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$, considered as a subspace of $X \times Y$. Show that $\Gamma(f)$ is homeomorphic to $X$.

Let $h : X \to \Gamma(f)$ be defined by $h(x) = (x, f(x))$. $h$ is continuous since each component function is continuous. Let $p : \Gamma(f) \to X$ be defined by $p(x, y) = x$. $p$ is continuous because it is the restriction to $\Gamma$ of the (continuous) projection $X \times Y \to X$.

Moreover, $(p \circ h)(x) = p(x, f(x)) = x$, and $(h \circ p)(x, y) = (x, f(x)) = (x, y)$ for $(x, y) \in \Gamma(f)$. Thus $h$ and $p$ are inverse bijections, so $X$ is homeomorphic to $\Gamma(f)$. 
5.(15) Parts (a) and (b) of this exercise shows that for general (non-metrizable) topological spaces a limit point of a subset \( A \) need not be the limit of a sequence of points in \( A \).

Let \( X = \mathbb{R} \) with the co-countable (or "countable complement") topology: a subset \( U \) is open iff \( U = \emptyset \) or \( \mathbb{R} - U \) is countable\(^3\). It is easy to check that this is indeed a topology on \( X \) (e.g., using the axioms for closed sets).

(a) Suppose \( A \subseteq X \) is an uncountable set (for instance, \( A = [0, 1] \)). Show that every point \( x \in X \) is a limit point of \( A \).

Hint: A subset of a countable set must be countable.

Let \( x \in X \). Let \( U \) be an open neighborhood of \( x \).

Then \( X - U \) is countable, so \((X -(U-\{x\})) = (X-U) \cup \{x\}\) is countable. Then \( A \not\subseteq (X -(U-\{x\})) \), so \( A \cap (U-\{x\}) \neq \emptyset \). Thus \( x \in A' \).

(b) Show that no sequence \((x_n)_{n=1}^{\infty}\) in \( X \) converges, unless for some \( N \geq 1 \), \( x_n = x_N \) for all \( n \geq N \).

Let \( x \in X \) and \( U = (X - \{x\} \cap \{x\}) \cup \{x\} \). Then \( X - U \subseteq \{x\} \cap \{x\} \) so \( X - U \) is countable. Thus \( U \) is an open neighborhood of \( x \). If \( \{x_n\}_{n=1}^{\infty} \) converges to \( x \), then \( \exists N \geq 1 \) such that \( x_n \in U \) for all \( n \geq N \), which implies \( x_n = x \) for all \( n \geq N \).

(c) Suppose \( X \) is a metric space, \( A \subseteq X \), and \( x \in A' \). Prove that there is a sequence \((x_n)_{n=1}^{\infty} \subseteq A \) that converges to \( x \).

For each \( n \geq 1 \), the open ball \( B(x, \frac{1}{n}) \) is an open nbhd of \( x \). Since \( x \in A' \), \( A \cap (B(x, \frac{1}{n}) - \{x\}) \neq \emptyset \), so we can choose \( x_n \in A \cap (B(x, \frac{1}{n}) - \{x\}) \). Claim \( \{x_n\}_{n=1}^{\infty} \) converges to \( x \): if \( U \) is an open nbhd of \( x \), then \( \exists \varepsilon > 0 \) with \( B(x, \varepsilon) \subseteq U \). Then \( \exists N \geq 1 \) with \( \frac{1}{N} < \varepsilon \), and then, for every \( n \geq N \), \( x_n \in B(x, \frac{1}{n}) \subseteq B(x, \frac{1}{N}) \subseteq B(x, \varepsilon) \subseteq U \).

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\(^3\) Recall, a set \( C \) is countable iff it is finite or there is a bijection \( N \to C \).