HIGHEST WEIGHT CLASSIFICATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE SPECIAL UNITARY GROUP

By Thomas Holtzworth

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Approved:

Michael Falk, Ph.D., Chair

Jim Swift, Ph.D.

William Schulz, Ph.D.

ABSTRACT

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Representation theory of the special unitary group, $\mathrm{SU}(n)$, has a fundamental role in theoretical physics. Therefore it is the purpose of this thesis to provide a detailed exposition of the highest weight classification of the irreducible representations of $\mathrm{SU}(n)$. This method is built from two objectives: provide a complete classification of all the irreducible representations, and build corresponding $\mathrm{SU}(n)$-modules to carry one of each.

The classification scheme is founded on the bijective correspondence between the representations of $\mathrm{SU}(n)$ and the finite-dimensional complex analytic representations of the special linear group of complex matrices, $\mathrm{SL}(n, \mathbb{C})$. These representations of $\mathrm{SL}(n, \mathbb{C})$ are accompanied by the presence of *weights*, resulting from the analytic nature of the representations. Ultimately, all such representations of $\mathrm{SL}(n, \mathbb{C})$ are uniquely identified by their *highest* weights, which additionally, are in a one to one correspondence with integer partitions of length less than or equal to $n-1$.

The construction phase utilizes the representation theory of the symmetric group, $\mathcal{S}_m$, and the general linear group of complex matrices, $\mathrm{GL}(n, \mathbb{C})$. Irreducible representation of $\mathcal{S}_m$ are identified by integer partitions, where the irreducible representation of $\mathrm{SL}(n, \mathbb{C})$ are found within the tensor power repre-

sentations of $\mathrm{GL}(n, \mathbb{C})$ on the mth tensor power of n-dimensional complex space. The symmetric group will be needed in constructing modules to carrying the tensor power representations of $\mathrm{GL}(n, \mathbb{C})$. Just as the symmetric and anti-symmetric subspaces of the tensor powers of n-dimensional complex space are the images of projection operators, realizations of the irreducible representation are constructed as the images of specific projection operators built from the integer partitions of the associated highest weights.

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Contents

Chapter 1

General representation theory

In this chapter general group representation theory will be introduced. Decomposition and classification theorems will be of primary importance as they will be necessary in the following chapters addressing representation theory for the symmetric group as well as tensor product representations of the group of complex invertible matrices.

Group representation theory can be described in the language of linear group actions or modules. Both are advantageous; however, the shape of this chapter will rest heavily on the latter. The main influences for the treatment here include the expositions given by Sagan [3] and Sternberg [4].

Unless otherwise stated, vector spaces are assumed to be complex and finite-dimensional.

1.1 Definitions and basic concepts

In this section the equivalent notions of group representations, matrix representations, and group modules will be defined.

If V and W are vector spaces, denote by $\mathrm{GL}(V)$ the group of invertible linear transformations on V, and denote by $\mathrm{Hom}_{\mathbb{C}}(V, W)$ the space of linear transformations from V to W. In addition, $\mathrm{GL}(n, \mathbb{C})$ is the group of complex $n \times n$ invertible matrices, and $M_n(\mathbb{C})$ is the space of all complex $n \times n$ matrices.

Definition 1.1.1. Let G be a group. A *representation* of G is a group homomorphism

$$\rho : G \to \mathrm{GL}(V),$$

where V is a vector space.

If $\rho : G \to \mathrm{GL}(V)$ is a representation of G, one says V *carries* the representation ρ.

Suppose $\dim V = n$. By fixing a basis for V, one identifies $\rho(g)$ with an $n \times n$ nonsingular complex matrix, leading to the following definition.

Definition 1.1.2. Let G be a group and n be a positive integer. Then, a *matrix representation* of G is a group homomorphism

$$X : G \to \mathrm{GL}(n, \mathbb{C}).$$

Definitions 1.1.1 and 1.1.2 are equivalent in the sense that, by use of a fixed basis of V, each gives rise to the other. Alternatively, the operation

$$v \mapsto \rho(g)v,$$

can be interpreted as multiplication of vectors $v \in V$ by group elements $g \in G$. This allows one to consider V as a left module over a ring, as described below.

Let G be a finite group. The *group algebra* of G (over $\mathbb{C}$), denoted as $\mathbb{C}[G]$, is the set of complex-valued functions on G, with its natural vector space structure and ring product defined by

$$(f_1 * f_2)(h) := \sum_{g \in G} f_1(g^{-1}h) f_2(g)$$

for all $g \in G$ and $f_1, f_2 \in \mathbb{C}[G]$. Consider G to be contained in $\mathbb{C}[G]$; then $*$ is the unique distributive, bilinear extension to $\mathbb{C}[G]$ of the group product on G. Let ε be the group identity element, and $f \in \mathbb{C}[G]$. Then

$$\varepsilon * f = f * \varepsilon = f.$$

Consequently, $\mathbb{C}[G]$ is a unital associative algebra over $\mathbb{C}$. Note $\mathbb{C}[G]$ is commutative if and only if G is abelian.

For simplicity, the use of $*$ will be suppressed. Furthermore, $f \in \mathbb{C}[G]$ will typically be expressed using the *formal sum*,

$$f = \sum_{g \in G} c_g g,$$

where $c_g = f(g) \in \mathbb{C}$. In the formal sum notation,

$$f_1 f_2 := (\sum_{g \in G} c_g g)(\sum_{h \in G} b_h h) = \sum_{h \in G}(\sum_{g \in G} c_g b_{g^{-1}h})h.$$

Let R be a ring with unity. A *left R-module* is an abelian group M endowed with scalar multiplication by elements of R,

$$(r, m) \mapsto rm,$$

such that, for all $m, n \in M$ and $r, s \in R$,

(1) $r(m + n) = rm + rn$,

(2) $(r + s)m = rm + sm$,

(3) $(sr)m = s(rm)$, and

(4) $1_R m = m$.

An *R-submodule* of an R-module M is an additive subgroup N of M satisfying $rx \in N$ for all $r \in R$, $x \in N$. For simplicity, a left $\mathbb{C}[G]$-module is called a *$G-$module*. If V is a $G-$module, then for each $g \in G$, the multiplication

$$v \mapsto gv$$

defines an element $\rho(g) \in \mathrm{GL}(V)$. The mapping ρ is representation of G. Conversely, if $\rho : G \to \mathrm{GL}(V)$ is a group representation, then the binary operation

$$G \times V \to V$$

defined by

$$(g, v) \mapsto \rho(g)v$$

endows V with a natural $G-$module structure. In this way, one has a bijective correspondence between representations of G and G-modules.

1.2 $G-$submodules and reducibility

Just like in other algebraic settings, notions of substructures and reducibility exist for group modules. In this section, these definitions will be extended to the setting of group representations. Additionally, the full reducibility of modules over finite groups will be presented through a fundamental result known as Maschke's Theorem.

For simplicity, the modifier "Hermitian" in "Hermitian inner product" will be omitted. In particular, if V has an inner product $\langle \cdot \mid \cdot \rangle$, then

(1) $\langle \lambda x \mid y \rangle = \overline{\lambda} \langle x \mid y \rangle$, and

(2) $\langle x \mid \lambda y \rangle = \lambda \langle x \mid y \rangle$

for all $\lambda \in \mathbb{C}$, $x, y \in V$. The notation $\perp$ denotes orthogonality relative to an inner product.

Definition 1.2.1. Let V be a $G-$module. A $G-$*submodule* of V is a subspace W such that

$$fw \in W$$

for all $f \in \mathbb{C}[G]$ and $w \in W$. A submodule W is *trivial* whenever $W = \{0\}$ or $W = V$.

If V carries the representation $\rho : G \to \mathrm{GL}(V)$, then a submodule W is said to be *invariant* under ρ. This terminology also extends to a matrix representation $X : G \to \mathrm{GL}(n, \mathbb{C})$ by considering $\mathbb{C}^n$ to carry the representation given by X.

Definition 1.2.2. A nonzero $G-$module V is *reducible* whenever it contains a non-trivial G-submodule. Otherwise, V is *irreducible*.

Moreover, if V is the (internal) direct sum of irreducible G-submodules, then V is said to be *completely reducible*.

One uses the same terminology for a representation $\rho : G \to \mathrm{GL}(V)$ by considering invariant subspaces of V instead of $G-$submodules. For example, ρ is irreducible whenever V has no non-trivial invariant subspaces. Likewise, this terminology is also used for a matrix representation $X : G \to \mathrm{GL}(n, \mathbb{C})$ by considering $\mathbb{C}^n$ to carry the representation given by X.

An inner product $\langle \cdot \mid \cdot \rangle$ on V is *invariant* under G if, for all $v, w \in V$ and $g \in G$,

$$\langle gv \mid gw \rangle = \langle v \mid w \rangle.$$

Lemma 1.2.3. *Let V be a $G-$module, W a G-submodule, and suppose $\langle \cdot \mid \cdot \rangle$ is an inner product on V invariant under G. Then $W^\perp$ is a $G-$submodule.*

Proof. Suppose W is a $G-$submodule. Let $v \in W^\perp$, $g \in G$, and $w \in W$. Then

$$\begin{aligned} \langle w \mid gv \rangle &= \langle g^{-1}w \mid g^{-1}(gv) \rangle \\ &= \langle g^{-1}w \mid (g^{-1}g)v \rangle \\ &= \langle g^{-1}w \mid v \rangle \\ &= 0, \end{aligned}$$

since $g^{-1}w \in W$. Hence $gv \in W^\perp$. Consequently, for all $f \in \mathbb{C}[G]$, $fv \in W^\perp$. ☐

Theorem 1.2.4 (Maschke's Theorem)**.** *Let G be a finite group, and V a nonzero $G-$module. Then V is completely reducible. In other words,*

$$V = W_1 \oplus W_2 \oplus ... \oplus W_n,$$

where each W_i is a nonzero irreducible $G-$submodule of V.

Proof. The proof will use induction on $n = \dim V$. If $n = 1$, then V is completely reducible, since V is irreducible itself.

So, suppose $\dim V = k > 1$, and that the result is true for all $G-$modules with dimension less than k. Suppose V is not already irreducible. Then from a basis $\{v_1, ..., v_k\}$ for V, one obtains a Hermitian inner product on V by setting

$$\langle v_i \mid v_j \rangle := \delta_{ij}.$$

Using $\langle \cdot | \cdot \rangle$, create a G-invariant Hermitian inner product by setting

$$\langle w \mid u \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gw \mid gu \rangle$$

for all $w, u \in V$. (It is straightforward to verify that $\langle \cdot | \cdot \rangle_G$ is Hermitian.)

Now, let W be a $G-$submodule of V. Note $W^\perp$ is also a G-submodule by Lemma 1.2.3. Furthermore, $\dim W < k$ and $\dim W^\perp < k$. Thus by the inductive hypothesis, W and $W^\perp$ are both completely reducible. Therefore $V = W \oplus W^\perp$ is completely reducible. ☐

1.3 Homomorphisms of $G-$Modules and Schur's Lemma

To obtain the full classification of irreducible G-modules for a group G, one will need to know when seemingly different modules are, in fact, the same. Homomorphisms will be needed to establish equivalences between various group modules and submodules. Here, these functions will be defined, and Schur's Lemma, a fundamental piece in the theory of group representations, will be stated and proved.

For an integer $n \geq 1$, denote the set $\{1, 2, ..., n\}$ by $[n]$.

Definition 1.3.1. Let V and W be G-modules for the group G. Then a complex-linear map

$$\phi : V \to W$$

is a $G-$*homomorphism* whenever

$$\phi(fv) = f\phi(v) \tag{1.3.1}$$

for all $v \in V$ and $f \in \mathbb{C}[G]$. The set of all such homomorphisms is denoted by $\mathrm{Hom}_G(V, W)$.

Note that ϕ is a G-homomorphism if and only if ϕ is a complex-linear transformation such that 1.3.1 holds for all $g \in G$.

Just like linear transformations, if ϕ is bijective then ϕ^{-1} is also a $G-$homomorphism. Therefore one has the following notion of equivalence.

Definition 1.3.2. Let V and W be $G-$modules for the group G. Suppose $\phi \in \mathrm{Hom}_G(V, W)$. If ϕ is bijective, then ϕ is a $G-$*isomorphism*. If so, then V and W are *isomorphic* as $G-$modules. However, if no such ϕ exists, then V and W are *inequivalent*.

Two representations $\rho : G \to \mathrm{GL}(V)$ and $\varrho : G \to \mathrm{GL}(W)$ for G are said to be *equivalent* whenever V and W are isomorphic as $G-$modules. Once more, there is analogous terminology for matrix representations.

For $\phi \in \mathrm{Hom}_G(V, W)$, the definition of its *kernel* and *image*, respectively denoted $\ker \phi$ and $\mathrm{im}\, \phi$, are the same as any linear transformation. In addition, the following usual algebraic result will be of use.

Proposition 1.3.3. *Let $\phi \in \mathrm{Hom}_G(V, W)$. Then $\ker \phi$ is a $G-$submodule of V, and $\mathrm{im}\, \phi$ is a G-submodule of W. Moreover, ϕ is injective if and only if $\ker \phi = \{0\}$.*

Corollary 1.3.4. *Suppose $\phi \in \mathrm{Hom}_G(V, W)$ is non-trivial.*

(1) *If V is irreducible, then ϕ is injective.*

(2) *If W is irreducible, then ϕ is surjective.*

Proof. By Proposition 1.3.3, $\ker \phi$ is a $G-$submodule of V. Thus $\ker \phi = \{0\}$ whenever V is irreducible, since ϕ is non-trivial. But, by Proposition 1.3.3, if $\ker \phi = \{0\}$, then ϕ is injective.

Now, suppose W is irreducible. Then, by Proposition 1.3.3, $\mathrm{im}\, \phi = W$. However, if $\mathrm{im}\, \phi = W$, then ϕ is surjective. □

Now one has Schur's Lemma.

Theorem 1.3.5 (Schur's Lemma)**.** *Let V and W be $G-$modules for the group G. Suppose $\phi \in \mathrm{Hom}_G(V, W)$ is non-trivial. If V and W are irreducible, then ϕ is a $G-$isomorphism.*

Proof. Suppose that ϕ is non-trivial. Then by Corollary 1.3.4, ϕ is bijection, since both V and W were assumed to be irreducible. Therefore ϕ is a $G-$isomorphism. □

Corollary 1.3.6. *Let $n \geq 1$, and $X : G \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible matrix representations for G. Let $A \in M_n(\mathbb{C})$, and suppose*

$$X(g)A = AX(g)$$

for all $g \in G$. Then for some $\lambda \in \mathbb{C}$, $A = \lambda I$.

Proof. Using $X : G \to \mathrm{GL}(n, \mathbb{C})$, one considers $\mathbb{C}^n$ a $G-$module by

$$gv = X(g)v,$$

where the right hand side is computed using matrix multiplication of $X(g)$ with the column vector v. Note $\mathbb{C}^n$ is an irreducible $G-$module since X is assumed to be an irreducible matrix representation.

The field is $\mathbb{C}$. Thus A has an eigenvalue $\lambda \in \mathbb{C}$, and hence $\ker(\lambda I - A) \neq \{0\}$. Since A commutes G,

$$(\lambda I - A)X(g) = X(g)(\lambda I - A)$$

for all $g \in G$. Consequently $(\lambda I - A)$ also commutes with G, making it a $G-$homomorphism on $\mathbb{C}^n$. However, $(\lambda I - A)$ is not a $G-$isomorphism since $\ker(\lambda I - A) \neq \{0\}$. By Schur's lemma, $(\lambda I - A) = 0$, and therefore $A = \lambda I$. $\square$

Corollary 1.3.7. *Let V and W be irreducible G- modules for G. If V and W are isomorphic as $G-$modules, then $\mathrm{Hom}_G(V, W)$ is one dimensional. Alternatively, if V and W are not isomorphic, then $\mathrm{Hom}_G(V, W) = \{0\}$.*

Proof. A quick application of Schur's lemma shows that $\mathrm{Hom}_G(V, W) = \{0\}$ if $V \ncong W$. So suppose otherwise.

First, consider the case when $W = V$ as $G-$modules and let $\phi \in \mathrm{Hom}_G(V, V)$. Since V is a finite dimensional complex vector space, there is a $\lambda \in \mathbb{C}$ such that

$$\ker(\lambda \operatorname{id}_V - \phi) \neq \{0\}.$$

Then by Schur's Lemma,

$$\phi = \lambda \operatorname{id}_V .$$

Therefore $\mathrm{Hom}_G(V, V)$ is one dimensional.

Consider the general case. Since V and W are isomorphic irreducible G- modules, pick some isomorphism $\phi \in \mathrm{Hom}_G(V, W)$, and let $\varphi \in \mathrm{Hom}_G(V, W)$. Then

$$\phi^{-1} \circ \varphi \in \mathrm{Hom}_G(V, V).$$

Thus for some $\lambda \in \mathbb{C}$,

$$\phi^{-1} \circ \varphi = \lambda \operatorname{id}_V .$$

Hence $\varphi = \lambda\phi$, and therefore $\mathrm{Hom}_G(V, W)$ is one dimensional. $\square$

1.4 Multiplicity and isotypic components

When a nonzero $G-$module V is completely reducible, it is possible that some of its submodules are isomorphic to one another. One can collect these isomorphic submodules together in the decomposition of V. With this in mind, write

$$V = V_1 \oplus V_2 \oplus ... \oplus V_m,$$

where, for each $i \in [m]$,

$$V_i = W_{i1} \oplus W_{i2} \oplus ... \oplus W_{in_i}$$

such that $W_{i1}, ..., W_{in_i}$ are isomorphic irreducible G-modules. Furthermore, if $i \neq l$, one insists that $W_{ij} \ncong W_{lk}$ for each $j \in [n_i]$ and $k \in [n_l]$.

Now, let W_i represent a common irreducible $G-$module equivalent to each the factors in V_i. Then V_i is called the *isotypic component associated with* W_i and n_i is its *multiplicity*. Finally, note that

$$\dim V = n_1 \dim W_1 + n_2 \dim W_2 + ... + n_m \dim W_m.$$

Proposition 1.4.1. *Let V be $G-$module that is completely reducible, and W be irreducible. Suppose V has only one isotypic component of multiplicity m with W being the corresponding irreducible $G-$module. Then*

$$V \cong \mathbb{C}^m \otimes W$$

as $G-$modules, where the action of G on $\mathbb{C}^m \otimes W$ is given by

$$g(u \otimes w) = u \otimes gw.$$

Proof. Under the hypothesis, $V = W_1 \oplus W_2 \oplus ... \oplus W_m$, where $W_i \cong W$ for each $i \in [m]$. Suppose

$$\phi_i : W \to W_i$$

is a corresponding isomorphism. Note that W_i is seen here as a subspace of V. Thus $V = \phi_1(W) \oplus \phi_2(W) \oplus ... \oplus \phi_m(W)$.

Let $\{u_i \mid\in [m]\}$ be the standard basis for $\mathbb{C}^m$, and $\{w_j \mid j \in [k]\}$ a basis for W, where $k = \dim W$. Then

$$\{u_i \otimes w_j \mid (i, j) \in [m] \times [k]\}$$

provides a basis for $\mathbb{C}^m \otimes W$. Now, $\mathbb{C}^m \otimes W$ becomes a $G-$module by setting

$$g(u \otimes w) := u \otimes gw$$

for all $u \in \mathbb{C}^m$ and $w \in W$.

Define $\Phi : \mathbb{C}^m \otimes W \to V$ by setting

$$\Phi(u_i \otimes w_j) := \phi_i(w_j)$$

for each $(i, j) \in [m] \times [k]$, and then extending by linearity to all of $\mathbb{C}^m \otimes W$. Note that Φ is a well defined linear transformation. Furthermore, since ϕ_i is an isomorphism,

$$\{\phi_i(w_j) \mid j \in [k]\}$$

is a basis for each $W_i \leq V$. Thus

$$\{\phi_i(w_j) \mid (i,j) \in [m] \times [k]\}$$

is a basis for $W_1 \oplus W_2 \oplus ... \oplus W_m = V$. This establishes Φ as a linear isomorphism.

Finally, let $g \in G$, and $j \in [k]$. Then for some collection of constants $\{g_{lj}\}$,

$$gw_j = \sum_{l=1}^{k} g_{lj} w_l.$$

Thus

$$\begin{aligned} g(\Phi(u_i \otimes w_j)) &= g\phi_i(w_j) \\ &= \phi_i(gw_j) \\ &= \phi_i\left(\sum_{l=1}^{k} g_{lj} w_l\right) \\ &= \sum_{l=1}^{k} g_{lj} \phi_i(w_l). \end{aligned}$$

Additionally,

$$\begin{aligned} \Phi(g(u_i \otimes w_j)) &= \Phi(u_i \otimes gw_j) \\ &= \Phi(u_i \otimes \left(\sum_{l=1}^{k} g_{lj} w_l\right)) \\ &= \Phi\left(\sum_{l=1}^{k} g_{lj} u_i \otimes w_l\right) \\ &= \sum_{l=1}^{k} g_{lj} \Phi(u_i \otimes w_l) \\ &= \sum_{l=1}^{k} g_{lj} \phi_i(w_l). \end{aligned}$$

Hence

$$g(\Phi(u_i \otimes w_j)) = \Phi(g(u_i \otimes w_j)),$$

and therefore $\mathbb{C}^m \otimes W \cong V$. $\square$

Corollary 1.4.2. *Let V be a completely reducible $G-$module. Suppose $\{W_i \mid i \in [m]\}$ is a collection of pairwise inequivalent irreducible $G-$modules, such that*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_m,$$

with each V_i being an isotypic component of multiplicity n_i associated to W_i. Then

$$V \cong \bigoplus_{i=1}^{m} \mathbb{C}^{n_i} \otimes W_i,$$

where the action of G on $\bigoplus_{i=1}^{m} \mathbb{C}^{n_i} \otimes W_i$ is given by

$$g(u_1 \otimes w_1 + ... + u_m \otimes w_m) = u_1 \otimes gw_1 + ... + u_m \otimes gw_m.$$

Proof. Repeat the proof of Proposition 1.4.1 to each isotypic component, V_i, then combine appropriately. □

Proposition 1.4.3. *Let V_1 and V_2 be two completely reducible G-modules having only one isotypic component each. Let m_1 and m_2 be the respective multiplicities with W_1 and W_2 being the corresponding irreducible $G-$modules. If $W_1 \cong W_2$, then*

$$\dim(\mathrm{Hom}_G(V_1, V_2)) = m_1 m_2.$$

Otherwise, $\dim(\mathrm{Hom}_G(V_1, V_2)) = \{0\}$.

Proof. Write

$$V_1 = \bigoplus_{l=1}^{m_1} W_{1l} \quad \text{and} \quad V_2 = \bigoplus_{l=1}^{m_2} W_{2l}$$

where $W_{il} \cong W_i$ for $i = 1, 2$. Note the following linear isomorphism, verifiable by considering the 'block form' of any given linear transformation from V_1 to V_2,

$$\mathrm{Hom}_{\mathbb{C}}(V_i, V_j) \cong \bigoplus_{l=1}^{m_1} \bigoplus_{k=1}^{m_2} \mathrm{Hom}_{\mathbb{C}}(W_{1l}, W_{2k}).$$

Likewise,

$$\mathrm{Hom}_G(V_i, V_j) \cong \bigoplus_{l=1}^{m_1} \bigoplus_{k=1}^{m_2} \mathrm{Hom}_G(W_{1l}, W_{2k})$$

since, for all l, k and $g \in G$,

$$\mathrm{Hom}_G(W_{1l}, W_{2k}) \leq \mathrm{Hom}_{\mathbb{C}}(W_{1l}, W_{2k}),$$

with $gW_{1l} = W_{1l}$, and $gW_{2k} = W_{2k}$.

Suppose now W_i and W_j are inequivalent. Then $\dim(\mathrm{Hom}_G(V_i, V_j)) = \{0\}$. Indeed, by Schur's lemma $\dim(\mathrm{Hom}_G(W_{1l}, W_{2k})) = \{0\}$ for each l and k. However, if W_i and W_j are isomorphic, then by Corollary 1.3.7, $\dim(\mathrm{Hom}_G(W_{1l}, W_{2k})) = 1$, for each l and k. Therefore

$$\dim(\mathrm{Hom}_G(V_i, V_j)) = \sum_{l=1}^{m_1}\sum_{k=1}^{m_2} \dim(\mathrm{Hom}_G(W_{1l}, W_{2k})) = m_1 m_2.$$

□

Corollary 1.4.4. *Let V be a completely reducible G-module, and $\{W_i \mid i \in [l]\}$ be a collection of pairwise distinct irreducible G-modules such that*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_l,$$

where each V_i is the isotypic component associated to W_i, has multiplicity m_i, and $\dim W_i = n_i$. Then for each $i \in [l]$, $\dim(\mathrm{Hom}_G(W_i, V)) = m_i$, and

$$\mathrm{Hom}_G(V_i, V_i) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i})$$

as rings.

Moreover,

$$\mathrm{Hom}_G(V, V) \cong \bigoplus_{i=1}^{l} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i}),$$

and in particular,

$$\dim(\mathrm{Hom}_G(V, V)) = m_1^2 + m_2^2 + ... + m_l^2.$$

Proof. By Proposition 1.4.3, $\dim(\mathrm{Hom}_G(W_i, V_j)) = m_i\delta_{ij}$. With this in mind, note that

$$\mathrm{Hom}_G(W_i, V) \cong \bigoplus_{j=1}^{l} \mathrm{Hom}_G(W_i, V_j).$$

Thus $\dim(\mathrm{Hom}_G(W_i, V)) = m_i$ for each $i \in [l]$.

A second use of Proposition 1.4.3 shows that

$$\dim(\mathrm{Hom}_G(V_i, V_j)) = m_i m_j \delta_{ij}.$$

Thus $\mathrm{Hom}_G(V_i, V_i) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i})$, where the congruence denotes a linear isomorphism. However, by choosing a bases $\mathcal{B}$ for V_i, one can verify that

$$[\phi \circ \varphi]_{\mathcal{B}} = [\phi]_{\mathcal{B}}[\varphi]_{\mathcal{B}}.$$

Consequently the isomorphism is also one of rings.

Finally, considering each V_i,

$$\mathrm{Hom}_G(V, V) \cong \bigoplus_{i=1}^{l} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i})$$

as rings, since

$$\mathrm{Hom}_G(V,V) \cong \bigoplus_{i,j=1}^{l} \mathrm{Hom}_G(V_i,V_j) = \bigoplus_{i=1}^{l} \mathrm{Hom}_G(V_i,V_i).$$

□

1.5 The classification of irreducible G-modules for finite groups

For a finite group G, the number of distinct irreducible G- modules is the same as the number of conjugacy classes in G. Furthermore, their exists a natural G-module structure on $\mathbb{C}[G]$, in which appears an isomorphic copy of each distinct irreducible G-module. Lastly, through the full decomposition of $\mathbb{C}[G]$ under this G-module structure, one will see that the sum of the squares of the dimensions of each irreducible G-module is equal to the order of the group.

All groups in this section are assumed to be finite.

1.5.1 The regular representation and the group algebra

Let X be a set and let

$$\mathbb{C}^X := \{f : X \to \mathbb{C} \mid |\mathrm{supp}(f)| < \infty\},$$

where $\mathrm{supp}(f) := \{x \in X \mid f(x) \neq 0\}$ is the (set-theoretic) *support* of f. For the following, consider $\mathbb{C}^X$ with its natural vector space structure.

The *standard basis* for $\mathbb{C}^X$ is the collection of functions $\{\delta_x \mid x \in X\}$, where for each $x \in X$

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}.$$

If $\mathrm{supp}(f) = \{x_1, x_2, ..., x_k\}$ for $f \in \mathbb{C}^X$, then f can be expressed in this basis as

$$f = \sum_{i=1}^{k} f(x_i)\delta_{x_i}.$$

Proposition 1.5.1. *If G acts on a set X, then $\mathbb{C}^X$ becomes a $G-$module by setting*

$$g(f) = f \circ g^{-1}.$$

Moreover, for each $x \in X$ and $g \in G$,

$$g(\delta_x) = \delta_{gx}.$$

Proof. Showing that $\mathbb{C}^X$ becomes a $G-$module under the defining action is simple. So the proof centers on the remaining claim.

Let $x \in X$, and $g \in G$. Then

$$g(\delta_x)(y) = \delta_x(g^{-1}y) = \begin{cases} 1 & \text{if } g^{-1}y = x \\ 0 & \text{if } g^{-1}y \neq x \end{cases}.$$

But $g^{-1}y = x$ if and only if $y = gx$. Therefore $g(\delta_x)(y) = \delta_{gx}(y)$ for all $y \in X$. $\square$

From Proposition 1.5.1, as a $G-$module, $\mathbb{C}^X$ carries the *permutation representation* associated with X.

Define an inner product on $\mathbb{C}^X$ by setting

$$\langle \delta_x \mid \delta_y \rangle = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

for each $x, y \in X$, and then extending by linearity to all of $\mathbb{C}^X$. This inner product is invariant under G, seen by the following proposition.

Proposition 1.5.2. *Let $f_1, f_2 \in \mathbb{C}^X$. For all $g \in G$*

$$\langle gf_1 \mid gf_2 \rangle = \langle f_1 \mid f_2 \rangle.$$

Proof. It will be enough to show that the result holds for the standard basis $\{\delta_x \mid x \in X\}$. Let $x, y \in X$, and $g \in G$. Then $gx = gy$ if and only if $x = y$. By definition

$$\langle \delta_{gx} \mid \delta_{gy} \rangle = \begin{cases} 1 & \text{if } gx = gy \\ 0 & \text{if } gx \neq gy \end{cases}.$$

Therefore $\langle \delta_{gx} \mid \delta_{gy} \rangle = \langle \delta_x \mid \delta_y \rangle$. $\square$

For $\mathbb{C}[G]$, the *regular representation* is the permutation representation associated with G induced from G acting on itself via left translation. Considering Proposition 1.5.1, let $g \in G$, and $f = \sum\limits_{h \in G} a_h h \in \mathbb{C}[G]$. Then

$$gf = \sum_{h \in G} a_h gh = \sum_{h \in G} a_{(g^{-1}h)} h.$$

This agrees with the defining action of G on $\mathbb{C}[G]$ outlined in Proposition 1.5.1 since gf is the function such that $(gf)(h) = f(g^{-1}h)$. However, under the formal sum characterization $a_{(g^{-1}h)} := f(g^{-1}h)$.

1.5.2 The decomposition of $\mathbb{C}[G]$

Here, the description of the decomposition of $\mathbb{C}[G]$ under the regular representation of G will be provided. The method to be used follows from an approach given by Sternberg [4].

Definition 1.5.3. Let V be a $G-$module. For each $(g, \alpha) \in G \times V^*$, define $g\alpha \in V^*$ by setting

$$(g\alpha)(v) = \alpha(g^{-1}v)$$

for all $v \in V$. With this action of G, V^* is the *dual $G-$module of V.*

Remark. It is straightforward to verify that the defining properties of a $G-$module are satisfied for V^* under this action of G.

Lemma 1.5.4. *Let V be a $G-$module. Then V is irreducible if and only if V^* is irreducible.*

Proof. Suppose V is irreducible, and W is a nonzero submodule of V^*. Then

$$\mathrm{Ann}^*(W) := \{v \in V \mid \alpha(v) = 0, \forall \alpha \in W\}$$

is a submodule of V Indeed, let $g \in G$, $\alpha \in W$, and suppose $v \in \mathrm{Ann}^*(W)$. Then

$$\alpha(gv) = (g^{-1}\alpha)(v) = 0$$

since $g^{-1}\alpha \in W$. Thus $gv \in \mathrm{Ann}^*(W)$.

Now, $W \neq \{0\}$ implies that $\mathrm{Ann}^*(W) \neq V$. As a result, $\mathrm{Ann}^*(W) = \{0\}$ since V is irreducible. However, $\mathrm{Ann}^*(W) = \{0\}$ implies that $W = V^*$.

Conversely, suppose V^* is irreducible, and W is a nonzero G-submodule of V. Then following the previous argument,

$$\mathrm{Ann}(W) := \{\alpha \in V^* \mid \alpha(v) = 0, \forall v \in W\}$$

is a proper submodule of V^*, i.e. $\mathrm{Ann}(W) \neq V^*$. Thus $\mathrm{Ann}(W) = \{0\}$, and hence $W = V$. Therefore V is irreducible as well. □

With the introduction of the dual module and Lemma 1.5.4, the time is now appropriate to justify the claim that each irreducible G-module can be embedded into $\mathbb{C}[G]$ as it carries the regular representation of G.

Lemma 1.5.5. *Let $\mathbb{C}[G]$ carry the regular representation of G, and V be an irreducible $G-$module. Then, for each $\alpha \in V^* \setminus \{0\}$, the map $f_\alpha : V \to \mathbb{C}[G]$ defined by*

$$f_\alpha(v)(g) = \alpha(g^{-1}v)$$

is an injective G-homomorphism.

Furthermore, let $\{\alpha_i \mid i \in [n]\}$ be a basis for V^*. Then the sum of subspaces of the images

$$\sum_{i=1}^{n} f_{\alpha_i}(V) \leq \mathbb{C}[G].$$

is a direct sum.

Proof. Let $\alpha \neq 0 \in V^*$. It is easy to verify that

$$v \mapsto f_\alpha(v)$$

is a complex-linear transformation by using the linearity of both α and the action of G on the module V. To see that it is also a $G-$ homomorphism consider the following.

Let $v \in V$, and $g \in G$. Then, for each $h \in G$,

$$\begin{aligned} f_\alpha(gv)(h) &= \alpha(h^{-1}(gv)) \\ &= \alpha((g^{-1}h)^{-1}v) \\ &= f_\alpha(v)(g^{-1}h) \\ &= (gf_\alpha(v))(h). \end{aligned}$$

Thus $f_\alpha \in \mathrm{Hom}_G(V, \mathbb{C}[G])$.

To show that f_α is injective, one will need the *submodule of V^* generated by α*, which is the subspace

$$\langle \alpha \rangle_G := \langle g\alpha \mid g \in G \rangle.$$

Since $\alpha \neq 0$, $\langle \alpha \rangle_G \neq \{0\}$. Then as a G-submodule, $\langle \alpha \rangle_G = V^*$. Indeed, by Lemma 1.5.4, V^* is an irreducible $G-$module since V is assumed to be irreducible.

Now, let $\beta \in V^*$, and suppose that $v \in \ker f_\alpha$. Then $\beta = \sum\limits_{g\in G} a_g g\alpha$ for some collection $\{a_g\}$, and $f_\alpha(v)$ is zero function on $\mathbb{C}[G]$. Thus for all $g \in G$,

$$\alpha(gv) = f_\alpha(v)(g^{-1}) = 0.$$

Hence

$$\begin{aligned} \beta(v) &= \sum_{g\in G} a_g g\alpha(v) \\ &= \sum_{g\in G} a_g \alpha(g^{-1}v) \\ &= 0. \end{aligned}$$

But this implies that $v \in \mathrm{Ann}^*(V^*) = \{0\}$. Therefore f_α is injective, since $\ker f_\alpha = \{0\}$.

Alternatively, suppose that $f_\alpha(w) = 0$ for all $w \in V$, and let $v \neq 0 \in V$. Then $\langle v \rangle_G$, the submodule of V generated by v, must be all of V, and for each $g \in G$,

$$\alpha(gv) = f_\alpha(v)(g^{-1}) = 0.$$

So, let $w \in V$, and note that, for some collection $\{a_g\}$, $w = \sum\limits_{g\in G} a_g gv$. Then

$$\begin{aligned} \alpha(w) &= \alpha\left(\sum_{g\in G} a_g gv\right) \\ &= \sum_{g\in G} a_g \alpha(gv) \\ &= 0. \end{aligned}$$

Thus $\alpha = 0$. Therefore the linear assignment $\alpha \mapsto f_\alpha$ is injective since its kernel is $\{0\} \leq V^*$.

Considering this result, let $\{\alpha_i \mid i \in [n]\}$ be a basis for V^*. Then $f_{\alpha_i}(V) \cap f_{\alpha_j}(V)$ is a submodule of both $f_{\alpha_i}(V)$ and $f_{\alpha_j}(V)$. Indeed, intersections of submodules are submodules themselves. So, if $f_{\alpha_i}(V) \cap f_{\alpha_j}(V) \neq \{0\}$, then

$$f_{\alpha_i}(V) = f_{\alpha_i}(V) \cap f_{\alpha_j}(V) = f_{\alpha_j}(V)$$

since both $f_{\alpha_i}(V)$ and $f_{\alpha_j}(V)$ are irreducible. Thus $f_{\alpha_i} \in \mathrm{Hom}_G(V, f_{\alpha_j}(V))$. However, $\mathrm{Hom}_G(V, f_{\alpha_j}(V))$ is one dimensional, which implies that $f_{\alpha_i} = cf_{\alpha_j}$ for some $c \in \mathbb{C}$. Therefore $\alpha_i = c\alpha_j$ by the injectivity of the linear assignment $\alpha \mapsto f_\alpha$. Hence $\alpha_i = \alpha_j$ since the α_i are linearly independent. In other words,

$$f_{\alpha_i}(V) \cap f_{\alpha_j}(V) = \{0\}$$

whenever $i \neq j$.

Pick any $j \in [n]$, and suppose

$$f_{\alpha_j}(V) \bigcap \left(\sum_{i \neq j} f_{\alpha_i}(V) \right) \neq \{0\}.$$

Then $f_{\alpha_j}(V) = \sum\limits_{i \neq j} f_{\alpha_i}(V)$ since sums of submodules are again submodules, and $f_{\alpha_j}(V)$ is irreducible. Pick another $k \neq j \in [n]$. Then

$$f_{\alpha_k}(V) \leq \sum_{i \neq j} f_{\alpha_i}(V) \leq f_{\alpha_j}(V).$$

Thus $f_{\alpha_k} \in \mathrm{Hom}_G(V, f_{\alpha_j}(V))$, and again $f_{\alpha_k} = cf_{\alpha_j}$ for some $c \in \mathbb{C}$. This time however, a clear contradiction has resulted: The previous statement implies that $\alpha_k = \alpha_j$. Therefore one has the direct sum,

$$\bigoplus_{i=1}^{n} f_{\alpha_i}(V) \leq \mathbb{C}[G].$$

□

Note that Lemma 1.5.5 has provided more than what was promised. Not only is there one copy of each irreducible G-module appearing in $\mathbb{C}[G]$, carrying the regular representation of G, there are at least the same number of distinct copies of a an irreducible G-module as its corresponding dimension.

Lemma 1.5.6. *Let V be a G-module, and $\langle \cdot \mid \cdot \rangle$ be an inner product invariant under G. Suppose that W_1 and W_2 are irreducible G-submodules. Then $W_1 \perp W_2$ whenever $W_1 \ncong W_2$ as G-modules.*

Proof. Since $\dim V < \infty$, W_1 and W_2 are closed (topologically) subspaces. Thus there exists P, an orthogonal projection from W_1 to W_2 Note, for each $w \in W_1$, $P(w)$ is the unique vector in W_2 such that

$$\langle u \mid P(w) \rangle = \langle u \mid w \rangle$$

for all $u \in W_2$. Furthermore, $W_1 \perp W_2$ if and only if $P = 0$. Indeed, for any orthonormal basis $\{u_i\}$ of W_2,

$$P(w) = \sum_{i=1}^{k} \langle u_i \mid w \rangle u_i,$$

where $k = \dim W_2$.

Suppose $W_1 \not\cong W_2$ as G- modules. Let $w \in W_1$, $u \in W_2$, and $g \in G$. Then $gP(w) \in W_2$, and

$$\begin{aligned} \langle u \mid gP(w) \rangle &= \langle g^{-1}u \mid P(w) \rangle \\ &= \langle g^{-1}u \mid w \rangle \\ &= \langle u \mid gw \rangle \end{aligned}$$

since $\langle \cdot \mid \cdot \rangle$ is invariant. But $P(gw)$ is the unique vector in W_2 such that

$$\langle u \mid P(gw) \rangle = \langle u \mid gw \rangle.$$

Thus $P(gw) = gP(w)$. Consequently

$$P \in \mathrm{Hom}_G(W_1, W_2) = \{0\}.$$

Hence $P = 0$, and therefore $W_1 \perp W_2$. □

With these two lemmas established, the focus turns to the decomposition of $\mathbb{C}[G]$ as a $G-$module under the regular representation. Note that $\dim \mathbb{C}[G] = |G|$. Thus one can apply Maschke's theorem for the following.

Theorem 1.5.7. *There are only finitely many distinct irreducible $G-$modules. Furthermore, let $\mathbb{C}[G]$ carry the regular representation of G, and $\{W_i \mid i \in [l]\}$ denote all the distinct irreducible G-modules. Suppose*

$$\mathbb{C}[G] = V_1 \oplus V_2 \oplus ... \oplus V_l,$$

where each V_i is the isotypic component associated to W_i, and has multiplicity m_i. Then for each $i \in [l]$,

$$m_i \geq \dim W_i.$$

Proof. Let V be an irreducible $G-$module with $\dim V = n$. Let $\{\alpha_i \in V^* \mid i \in [n]\}$ be a basis for V^*, and each f_{α_i} be the injective $G-$homomorphism from Lemma 1.5.5. Recall that $f_{\alpha_i}(V)$ is an irreducible $G-$submodule of $\mathbb{C}[G]$. So, $\mathbb{C}[G]$, a $G-$module under the regular representation of G, has an isomorphic copy of the irreducible V.

Now, suppose that $\{W_i \mid i \in [l]\}$ is a collection of pairwise distinct $G-$modules such that

$$\mathbb{C}[G] = V_1 \oplus V_2 \oplus ... \oplus V_l,$$

where each V_i is the isotypic component associated to W_i, and has multiplicity m_i. Note that $\mathbb{C}[G]$ naturally has an inner product $\langle \cdot \mid \cdot \rangle$ invariant under G, described by $\langle g \mid h \rangle = \delta_{gh}$. Let

$i \in [l]$ and $j \in [n]$ and write

$$V_i = W_{i1} \oplus W_{i2} \oplus \ldots \oplus W_{im_i}$$

where each $W_{ik} \cong W_i$. Using Lemma 1.5.6, if $V \not\cong W_i$, then $f_{\alpha_j}(V) \perp W_{ik}$ for all $k \in [m_i]$. Thus $f_{\alpha_j}(V) \perp V_i$ as well. With this reasoning, if $V \not\cong W_i$ for all $i \in [l]$, then $f_{\alpha_j}(V) \perp \mathbb{C}[G]$. Indeed, $f_{\alpha_j}(V) \perp V_i$ for each $i \in [l]$. However, this can't happen since $\mathbb{C}[G]^{\perp} = \{0\}$. Therefore for some $i \in [l]$, $V \cong W_i$. In particular, there can only be finitely many distinct irreducible $G-$modules for G.

Finally, let V_i be the isotypic component associated to W_i such that $V \cong W_i$. Then since $f_{\alpha_j}(V) \leq V_i$, for each $j \in [n]$, Lemma 1.5.5 guarantees that one has the direct sum

$$\bigoplus_{j=1}^{n} f_{\alpha_j}(V) \leq V_i$$

Therefore $(\dim W_i)^2 = n^2 \leq m_i \dim W_i$, and hence $\dim W_i \leq m_i$. □

In remaining portion of this subsection, a full description of the decomposition of $\mathbb{C}[G]$ under the regular representation of G will be provided by showing that the number isomorphic copies of each irreducible G-modules appearing in the decomposition of $\mathbb{C}[G]$ is, in fact, equal to the dimension of that particular irreducible G-module. Through this, the formula relating the order of the group to the squared values of the dimensions of the irreducible G-modules will be apparent.

Let $\mathbb{C}[G \times G]$ be the group algebra of the direct product of G with itself. Let $\mathbb{C}[G \times G]$ carry the permutation representation induced from the action of G on $G \times G$, given by

$$(h_1, h_2) \to (gh_1, gh_2).$$

Also, consider $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G])$, the space of linear operators on $\mathbb{C}[G]$, as a $G-$module under the group action

$$(g, \varphi) \to g \circ \varphi \circ g^{-1}.$$

The point here is that claiming the operator ϕ is a $G-$homomorphism is equivalent to claiming that ϕ is fixed under this action, i.e. $g \circ \phi \circ g^{-1} = \phi$ for all $g \in G$.

Now, let $K \in \mathbb{C}[G \times G]$, and $F \in \mathbb{C}[G]$. One can create a unique linear operator on $\mathbb{C}[G]$ using K. Indeed, let

$$\Xi_K F \in \mathbb{C}[G],$$

be given by

$$(\Xi_K F)(g) = \sum_{h \in G} K(g, h) F(h).$$

Here standard function notation was used instead of the formal sum notation for simplicity.

Lemma 1.5.8. *The function* $\Xi : \mathbb{C}[G \times G] \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G])$ *is a* $G-$*isomorphism. In particular,* $\mathbb{C}[G \times G] \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G])$ *as* $G-$*modules.*

Proof. To start, it is clear that Ξ is well defined and linear. So the real effort comes from showing that Ξ is $G-$ homomorphism, and additionally, a bijection. Since, this is in the realm of finite dimensional vector spaces, one can show Ξ is a bijection by just verifying that $\ker \Xi = 0$.

With that said, suppose that $K \in \ker \Xi$, and let and $F = \delta_g \in \mathbb{C}[G]$. Then for all $h \in G$,

$$\begin{aligned} 0 &= (\Xi_K F)(h) \\ &= \sum_{h' \in G} K(h,h')F(h') \\ &= K(h,g). \end{aligned}$$

Therefore $K = 0$,, and hence Ξ is a bijection. Now, once it is shown that Ξ is a $G-$ homomorphism, the proof will be complete.

So, let $g \in G$ and $K \in \mathbb{C}[G \times G]$. Then for $F \in \mathbb{C}[G]$,

$$\begin{aligned} (\Xi_{gK} F)(h) &= \sum_{h' \in G} (gK)(h,h')F(h') \\ &= \sum_{h' \in G} K(g^{-1}h, g^{-1}h')F(h') \\ &= \sum_{h' \in G} K(g^{-1}h, h')F(gh'). \end{aligned}$$

Alongside this, $(g \circ \Xi_K \circ g^{-1})(F) \in \mathbb{C}[G]$ is given by

$$\begin{aligned} \left((g \circ \Xi_K \circ g^{-1})F\right)(h) &= \left(g(\Xi_K g^{-1}F)\right)(h) \\ &= (\Xi_K g^{-1}F)(g^{-1}h) \\ &= \sum_{h' \in G} K(g^{-1}h, h')(g^{-1}F)(h') \\ &= \sum_{h' \in G} K(g^{-1}h, h')F(gh'). \end{aligned}$$

Therefore $\Xi_{gK} = g \circ \Xi_K \circ g^{-1}$, and hence Ξ is a $G-$ homomorphism. □

From this result, one can describe operators in $\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ in the setting of $\mathbb{C}[G \times G]$. To be exact, $\Xi_K \in \mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ if and only if

$$\Xi_{gK} = g \circ \Xi_K \circ g^{-1} = \Xi_K$$

for all $g \in G$. Since Ξ is isomorphism, this amounts to $gK = K$ for each $g \in G$. With this in mind, the following lemma gives the necessary and sufficient conditions for a function to be fixed under the action of a permutation representation. It also gives the dimension of the subspace of such functions in terms of the number of orbits of G in X.

Lemma 1.5.9. *Let $f \in \mathbb{C}^X$, where $\mathbb{C}^X$ carries the permutation representation of some action of G on the finite set X. Then $gf = f$ for all $g \in G$ if and only if f is constant on the orbits of G in X. Furthermore, the dimension of the subspace of functions fixed under the permutation representation is the number of orbits in X.*

Proof. Suppose f is constant on the orbits of G in X. Then for any $x \in X$,

$$\begin{aligned}(gf)(x) &= f(g^{-1}x)\\ &= f(x)\end{aligned}$$

since $g^{-1}x$ and x lie in the same orbit.

Conversely, suppose that $gf = f$ for all $g \in G$, and let $\mathcal{O}_x$ denote the orbit of x under G. For another $y \in \mathcal{O}_x$, let $h \in G$ be given such that $y = hx$. Then

$$f(y) = f(hx) = (h^{-1}f)(x) = f(x).$$

Therefore f is constant on orbits.

Now the collection of functions fixed under the permutation representation forms a subspace in $\mathbb{C}^X$. In fact, it is also a G-submodule, whose irreducible components are all equivalent to the trivial module. Using the previous result, one can determine the dimension of this submodule by determining the dimension of the subspace of functions that are constant on orbits. This number is simply the number of orbits themselves. To see this, let $\{\mathcal{O}_i \mid i \in [k]\}$ denote all the orbits in X, where for any one $\mathcal{O}_i$, set

$$\chi_i := \sum_{x \in \mathcal{O}_i} \delta_x.$$

Then the collection $\{\chi_i \mid i \in [k]\}$ is a basis for this subspace of functions. Therefore the number of orbits is the corresponding dimension. With this, the number of orbits is also the dimension of the $G-$submodule of functions fixed under the permutation representation. $\square$

Proposition 1.5.10. *Let $\mathbb{C}[G]$ carry the regular representation of G, and $\{W_i \mid i \in [l]\}$ denote a complete set of the distinct irreducible $G-$modules. Then*

$$\dim(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])) = |G|.$$

In particular $|G| = m_1^2 + m_2^2 + ... + m_l^2$, where each m_i is the multiplicity of the isotypic component in the decomposition of $\mathbb{C}[G]$ associated to W_i.

Proof. Using $\Xi : \mathbb{C}[G \times G] \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G])$, the $G-$isomorphism used in Lemma 1.5.8, one can determine the dimension of $\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ in the setting of $\mathbb{C}[G \times G]$. Indeed, by Lemma 1.5.9, $\phi \in \mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ if and only if $\Xi^{-1}(\phi) \in \mathbb{C}[G \times G]$ is constant on the orbits of G in $G \times G$. Furthermore,

$$\dim(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])) = \text{“the number of orbits of } G \text{ in } G \times G.\text{”}$$

The claim is that $\{(\varepsilon, g) \mid g \in G\}$ denotes a complete set of representatives for these orbits in $G \times G$. Let $(g, h) \in G \times G$. Then $g(\varepsilon, g^{-1}h) = (g, h)$. Thus every element in $G \times G$ is in an orbit determined by elements in $\{(\varepsilon, g) \mid g \in G\}$.

Now, suppose that there is some pair h and h' in G such that for some $g \in G$, $g(\varepsilon, h) = (\varepsilon, h')$. Then $\varepsilon = g\varepsilon$, and $h' = gh$. But, clearly this means that $g = \varepsilon$. Thus $(\varepsilon, h) = (\varepsilon, h')$.

Consequently the number such orbits is equal to $|G|$. Therefore

$$\dim(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])) = |G|.$$

Finally, if $\{W_i \mid i \in [l]\}$ denotes a complete set of the distinct irreducible $G-$modules, then by Corollary 1.4.3,

$$|G| = \dim(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]) = m_1^2 + m_2^2 + ... + m_l^2,$$

where each m_i is the multiplicity of the isotypic component in the decomposition of $\mathbb{C}[G]$ associated to W_i. □

Corollary 1.5.11. *Let $\{W_i \mid i \in [l]\}$ denote a complete set of the distinct irreducible $G-$modules, where $n_i = \dim W_i$ for each $i \in [l]$, and m_i be the multiplicity of the isotypic component in the decomposition of $\mathbb{C}[G]$ associated to W_i. Then $m_i = n_i$ for each $i \in [l]$, and*

$$|G| = n_1^2 + n_2^2 + ... + n_l^2.$$

Proof. By Proposition 1.5.10, $|G| = m_1^2 + m_2^2 + ... + m_l^2$, where each m_i is the multiplicity of the isotypic component in the decomposition of $\mathbb{C}[G]$ associated to W_i. However, using Theorem 1.5.7, one sees that $m_i \geq n_i$ for each $i \in [l]$. Now if, for some $i \in [l]$, $m_i > n_i$, then

$$|G| = m_1^2 + m_2^2 + ... + m_l^2 > m_1 n_1 + m_2 n_2 + ... + m_l n_l.$$

However, this is impossible since, as a direct sum of all the isotypic components,

$$|G| = m_1 n_1 + m_2 n_2 + ... + m_l n_l.$$

Therefore $m_i = n_i$ for each $i \in [l]$, and $|G| = n_1^2 + n_2^2 + ... + n_l^2$. □

1.5.3 The number of distinct irreducible group modules

The chapter concludes with the property that the number of distinct irreducible $G-$modules is just the number of conjugacy classes in G. Here, the justification to be given is modeled from a method used by Sagan [3].

Recall that under Maschke's theorem, any G-module is completely reducible. So, for each $l \geq 1$, let

$$Z_l(\mathbb{C}) = \{D \in M_l(\mathbb{C}) \mid D = \mathrm{diag}(d_1, d_2, ..., d_l)\},$$

and

$$\mathcal{Z}(\mathrm{Hom}_G(V, V)) = \{\phi \in \mathrm{Hom}_G(V) \mid \varphi \circ \phi = \phi \circ \varphi,\ \forall \varphi \in \mathrm{Hom}_G(V, V)\}.$$

Proposition 1.5.12. *Let V be a G-module, and $\{W_i \mid i \in [l]\}$ be a collection of pairwise distinct irreducible G-modules such that*

$$V = V_1 \oplus V_2 \oplus ... \oplus V_l,$$

where each $i \in [l]$, V_i is the isotypic component associated to W_i, and has multiplicity m_i. Then

(1) $\mathcal{Z}(\mathrm{Hom}_G(V,V)) \cong Z_l(\mathbb{C})$,

(2) $\dim(\mathcal{Z}(\mathrm{Hom}_G(V,V))) = l$.

Proof. Let $i \in [l]$. Then from Corollary 1.4.2, is the following isomorphism of rings

$$\mathrm{Hom}_G(V_i, V_i) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i}) \cong M_{m_i}(\mathbb{C}).$$

Now, it is well known that

$$\mathcal{Z}(M_{m_i}(\mathbb{C})) := \{A \in M_{m_i}(\mathbb{C}) \mid AB = BA, \forall B \in M_{m_i}(\mathbb{C})\} = \{\lambda I_{m_i} \mid \lambda \in \mathbb{C}\}.$$

(The notation I_{m_i} will be used to distinguish the identity matrix in different dimensions.) Thus

$$\mathcal{Z}(\mathrm{Hom}_G(V_i)) \cong \{\lambda I_{m_i} \mid \lambda \in \mathbb{C}\} \cong \mathbb{C}$$

for each $i \in l$. Therefore if $\mathbb{C}^l$, as a ring, is the lth direct product of $\mathbb{C}$, then

$$\mathcal{Z}(\mathrm{Hom}_G(V,V)) \cong \bigoplus_{i=1}^{l} \mathcal{Z}(\mathrm{Hom}_G(V_i)) \cong \mathbb{C}^l \cong Z_l(\mathbb{C}),$$

and hence $\dim(\mathcal{Z}(\mathrm{Hom}_G(V,V))) = l$. $\square$

The following is a direct application of Proposition 1.5.12 to the context of $\mathbb{C}[G]$.

Corollary 1.5.13. *Let $\mathbb{C}[G]$ carry the regular representation of G, and $\{W_i \mid i \in [l]\}$ denote a complete collection of distinct irreducible G-modules. Then*

(1) $\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])) \cong Z_l(\mathbb{C})$,

(2) $\dim(\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))) = l$.

This is useful since one can compute the total number distinct irreducible $G-$modules by determining the dimension of the center of $\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$. However, it turns out that one can further reduce the problem to the setting of $\mathbb{C}[G]$. To be exact, it will be shown that

$$\dim(\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))) = \dim(\mathcal{Z}(\mathbb{C}[G])).$$

Consider the $G-$isomorphism, $\Xi : \mathbb{C}[G \times G] \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G])$, introduced earlier along with the collection $\{\chi_g \in \mathbb{C}[G \times G] \mid g \in G\}$, where, inspired by Lemma 1.5.9,

$$\chi_g := \chi_{(\varepsilon, g)} = \sum_{h \in G} (h, hg)$$

for each $g \in G$. Note that $\{\chi_g \in \mathbb{C}[G \times G] \mid g \in G\}$ is, in fact, a basis for the $G-$submodule of functions in $\mathbb{C}[G \times G]$ fixed under the action of G. Now, set $\Xi_g := \Xi_{\chi_g}$ for each $g \in G$. Then $\{\Xi_g \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G]) \mid g \in G\}$ is a basis for $\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$. For each $f \in \mathbb{C}[G]$, set

$$\Xi_f = \sum_{g \in G} a_g \Xi_g,$$

where $f = \sum_{g \in G} a_g g$. Now consider the following proposition.

Proposition 1.5.14. *The map $\psi : \mathbb{C}[G] \to \mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ defined by $\psi(f) = \Xi_f$ is a linear isomorphism such that*

$$\psi(f_1 f_2) = \psi(f_1) \circ \psi(f_2)$$

for all $f_1, f_2 \in \mathbb{C}[G]$. Furthermore, $\mathcal{Z}(\mathbb{C}[G]) \cong \mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))$ as rings. In particular,

$$\dim(\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))) = \dim(\mathcal{Z}(\mathbb{C}[G])).$$

Proof. First, ψ is easily seen to be well defined and linear. So suppose that $f \in \ker \psi$. If $f = \sum_{g \in G} a_g g$, then

$$\begin{aligned} 0 &= \psi(f) \\ &= \sum_{g \in G} a_g \Xi_g. \end{aligned}$$

This means that, for all $g \in G$, $a_g = 0$ since $\{\Xi_g \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], \mathbb{C}[G]) \mid g \in G\}$ is a linearly independent set. Thus $f = 0$. Therefore ψ is injective, and hence a linear isomorphism.

Now let $g \in G$,, and $f = \sum_{g \in G} a_g g$ be in $\mathbb{C}[G]$. Note that, for all $c, h \in G$,

$$\chi_g(c, h) = \begin{cases} 1 & \text{if } h = cg \\ 0 & \text{if otherwise.} \end{cases}$$

Indeed, (c, h) is in the orbit of (ε, g) in $G \times G$ if and only if $c(\varepsilon, c^{-1}h) = (c, h)$ with $c^{-1}h = g$. Using this result let $c \in G$. Then

$$\begin{aligned} (\Xi_g f)(c) &= \sum_{h \in G} \chi_g(c, h) f(h) \\ &= f(cg). \end{aligned}$$

Let $h \in G$, and consider

$$\begin{aligned} (\Xi_{gh} f)(c) &= f(c(gh)) \\ &= f((cg)h) \\ &= (\Xi_h f)(cg) \\ &= (\Xi_g(\Xi_h f))(c) \\ &= ((\Xi_g \circ \Xi_h) f)(c). \end{aligned}$$

Thus $\psi(gh) = \psi(g) \circ \psi(h)$ for all basis elements $g, h \in \mathbb{C}[G]$. So, if $f_1 = \sum_{g \in G} a_g g$ and

$f_2 = \sum_{g \in G} b_g g$, then

$$\begin{aligned}
\psi(f_1 f_2) &= \psi\Big(\sum_{(g,h) \in G \times G} a_g b_h gh\Big) \\
&= \sum_{(g,h) \in G \times G} a_g b_h \psi(gh) \\
&= \sum_{(g,h) \in G \times G} a_g b_h \psi(g) \circ \psi(h) \\
&= \psi(f_1) \circ \psi(f_2).
\end{aligned}$$

Therefore ψ preserves multiplication as well. From this, $\mathbb{C}[G] \cong \mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])$ as rings as well as vector spaces. Finally, this implies that

$$\psi(\mathcal{Z}(\mathbb{C}[G])) = \mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G])),$$

and thus $\dim(\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))) = \dim(\mathcal{Z}(\mathbb{C}[G]))$. □

Lemma 1.5.15. *Let $f \in \mathbb{C}[G]$. Then $f \in \mathcal{Z}(\mathbb{C}[G])$ if and only if $gfg^{-1} = f$ for all $g \in G$. Furthermore, let s be the number of conjugacy classes in G. Then the following is a basis for $\mathcal{Z}(\mathbb{C}[G])$*

$$\{\chi_i \in \mathbb{C}[G] \mid i \in [s]\}$$

where, for each $i \in [s]$, C_i denotes the ith conjugacy class, and

$$\chi_i := \sum_{g \in C_i} g.$$

Proof. Let $f \in \mathbb{C}[G]$. Clearly $f \in \mathcal{Z}(\mathbb{C}[G])$ implies that $gfg^{-1} = f$ for all $g \in G$ since each g gives a basis element of $\mathbb{C}[G]$, and $gfg^{-1} = g$ holds if and only if $gf = fg$.

So, suppose that $gfg^{-1} = f$ for all $g \in G$, and let $d = \sum_{g \in G} a_g g$ be in $\mathbb{C}[G]$. Since $gf = fg$ for each $g \in G$,

$$\begin{aligned}
fd &= f\left(\sum_{g \in G} a_g g\right) \\
&= \sum_{g \in G} a_g f g \\
&= \sum_{g \in G} a_g g f \\
&= \left(\sum_{g \in G} a_g g\right) f \\
&= df.
\end{aligned}$$

Now considering Lemma 1.5.9, note that $\mathbb{C}[G]$ also carries the permutation representation corresponding to the action of conjugation by G on itself. Thus $\mathcal{Z}(\mathbb{C}[G])$ is the subspace of functions fixed under this action . Furthermore, the conjugacy classes in G are, by definition, the orbits under conjugation. Hence $\mathcal{Z}(\mathbb{C}[G])$ is also the subspace of functions constant on conjugacy classes of G. Now, the collection $\{\chi_i \in \mathbb{C}[G] \mid i \in [s]\}$ was defined so that it would be a basis for such functions. Therefore it is a basis for $\mathcal{Z}(\mathbb{C}[G])$ as well. $\square$

Theorem 1.5.16. *Let G be a finite group. Then the number of distinct irreducible $G-$modules is the number of conjugacy classes is in G.*

Proof. By Lemma 1.5.15, the dimension of $\mathcal{Z}(\mathbb{C}[G])$ is equal to the number of conjugacy classes in G. Furthermore, considering Proposition 1.5.14, the dimension of

$$\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))$$

is also equal to the number of conjugacy classes in G. Finally by Corollary 1.5.13, the number of distinct irreducible G-modules is equal to the dimension of $\mathcal{Z}(\mathrm{Hom}_G(\mathbb{C}[G], \mathbb{C}[G]))$. Therefore the result follows $\square$

Chapter 2

Representations of the symmetric group

The objective of this chapter is to build each irreducible representation of the symmetric group. The corresponding group modules will then be realized in the setting of $\mathbb{C}[\mathcal{S}_n]$. These modules will be vital in the construction of the irreducible tensor representations of the group of complex invertible matrices of specified degree. The treatment here follows mainly the exposition given by Sagan [3].

For reference, the *symmetric group* $\mathcal{S}_n$ is the collection of all bijections from $\{1, 2, ..., n\}$ to itself, using function composition as the group product. Elements $\sigma \in \mathcal{S}_n$ are called *permutations*, and for any two permutations σ and τ, juxtaposition will be used to denote their product, i.e.

$$\sigma\tau \equiv \sigma \circ \tau.$$

Furthermore, permutations will be displayed as products of disjoint *m-cycles*. See Chapter 3 in Rotman [2] for a thorough treatment of cycle notation.

2.1 Cycle type and integer partitions

As a finite group, $\mathcal{S}_n$ has the same number of distinct irreducible group representations as conjugacy classes. Furthermore, two permutations in $\mathcal{S}_n$ are conjugate if and only if they have the same structure known as cycle type, which can itself be uniquely characterized using integer partitions. This section defines these terms and outlines the correspondence between the conjugacy classes of $\mathcal{S}_n$ and the integer partitions of n. This is appropriate since each integer partition will determine a unique irreducible $\mathcal{S}_n$-module. Consequently, a complete set of irreducible group representation of $\mathcal{S}_n$ will result.

The notion of *cycle type* characterizes a permutation solely in terms of the number of its k-cycles. Each possible cycle type will be given using an integer partition.

Definition 2.1.1. Let $n \in \mathbb{N}$. A *partition* of n, denoted as $\lambda \vdash n$, is a sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$$

such that, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l$ and $\sum_{j=1}^{l} \lambda_j = n$.

Let $0 \leq m_k \leq n$ be the number of k-cycles in a permutation $\sigma \in \mathcal{S}_n$. Then

$$\lambda_\sigma := (\underbrace{n, n, ..., n}_{m_n \text{ times}}, \underbrace{n-1, n-1, ..., n-1}_{m_{n-1} \text{ times}}, ..., \underbrace{1, 1, ..., 1}_{m_1 \text{ times}}) \tag{2.1.1}$$

is the partition of n corresponding to the cycle type of σ. Note that if there are no k-cycles of a certain length k in σ, i.e. $m_k = 0$, then the number k does not appear in the partition. For example, the first entry in λ_σ will be the the length of the largest cycle in σ. Furthermore, by the method of construction in 2.1.1, it should be clear that different cycle types define distinct integer partitions.

Example 2.1.2. Let $n = 4$. Then $\lambda_1 = (4)$, $\lambda_2 = (1, 1, 1, 1)$, and $\lambda_3 = (2, 2)$ are the integer partitions corresponding to $\sigma_1 = (1\,2\,3\,4)$, $\sigma_2 = (1)(2)(3)(4)$, and $\sigma_3 = (1\,2)(3\,4)$, respectively.

To conclude this section, the following proposition summarizes the points just presented, a proof of which can be found in Sagan [3].

Proposition 2.1.3. *Let σ and τ be two permutations in $\mathcal{S}_n$. Then σ and τ have the same cycle type if and only if they are conjugates. In particular, the number of conjugacy classes, cycle types, and partitions of n are equal to one another. Moreover, the number of permutations of a given cycle type is*

$$\frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! ... n^{m_n} m_n!}.$$

2.2 Tableaux, tabloids and permutation modules

The purpose of this section is to build a family of $\mathcal{S}_n$-modules in connection with each integer partition. The method is summarized by the following. First, assemble a collection of objects for each partition. Afterward, define an action of $\mathcal{S}_n$ on this collection. Then finally, form the associated permutation representation. These modules will, in general, be reducible; however, within each one, exists the desired irreducible.

Definition 2.2.1. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$. The *Ferrers diagram*, or *shape*, of λ is an array of n boxes or cells having l left justified rows with the ith row having λ_i cells for $1 \leq i \leq l$.

One can also characterize a particular shape of λ using entry variables (or indeterminates) of an $n \times n$ matrix. If the entries are given by

$$\{x_{i,j} \mid (i, j) \in [n] \times [n]\},$$

then the shape of λ would by the subset

$$\{x_{i,j} \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\}.$$

Definition 2.2.2. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$. A *Young tableau of shape* λ, is a bijective assignment

$$t : \{x_{i,j} \mid 1 \leq i \leq l,\ 1 \leq j \leq \lambda_i\} \to [n].$$

The set of all λ-tableaux is denoted
$$\mathrm{Tab}(\lambda).$$

A Young tableau of shape λ is also called a λ-*tableau*. Note that there are $n!$ different tableaux of shape λ. Also, if $t : x_{i,j} \to k$, one writes $t_{i,j} = k$. This appeals to the idea that the integer k has been placed into the cell of position (i, j) in the Ferrers diagram.

Definition 2.2.3. Let t and u be two λ-tableaux. If

$$\{t_{i,j} \mid j \in [\lambda_i]\} = \{u_{i,j} \mid j \in [\lambda_i]\},$$

for all $i \in [l]$, then t and u are considered *row equivalent*. This is an equivalence relation on $\mathrm{Tab}(\lambda)$, so one writes $t \sim u$ whenever t and u are row equivalent. The equivalence class

$$\{t\} = \{u \in \mathrm{Tab}(\lambda) \mid u \sim t\},$$

is a *tabloid of shape* λ, or a λ-*tabloid*.

Proposition 2.2.4. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, and t be a λ-tableau. Then, $|\{t\}| = \lambda_1!\lambda_2!...\lambda_l!$. Therefore, the number of all λ-tabloids is*

$$\frac{n!}{\lambda_1!\lambda_2!...\lambda_l!}.$$

For each $\sigma \in \mathcal{S}_n$ and $t \in \mathrm{Tab}(\lambda)$, define $\sigma t \in \mathrm{Tab}(\lambda)$ by

$$(\sigma t)_{i,j} = \sigma(t_{i,j}).$$

Theorem 2.2.5. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, and t, u be two λ-tableaux. Then for any $\sigma \in \mathcal{S}_n$,*
$$\sigma t \sim \sigma u$$
whenever $t \sim u$.

Proof. Let $\sigma \in \mathcal{S}_n$, and $i \in [l]$. Suppose t and u are row equivalent. Then

$$\{t_{i,j} \mid j \in [\lambda_i]\} = \{u_{i,j} \mid j \in [\lambda_i]\}.$$

Thus $\{\sigma(t_{i,j}) \mid j \in [\lambda_i]\} = \{\sigma(u_{i,j}) \mid j \in [\lambda_i]\}$. Note that $(\sigma t)_{i,j} = \sigma(t_{i,j})$, and $(\sigma u)_{i,j} = \sigma(u_{i,j})$. Consequently
$$\{(\sigma t)_{i,j} \mid j \in [\lambda_i]\} = \{(\sigma u)_{i,j} \mid j \in [\lambda_i]\}.$$
Therefore $\sigma t \sim \sigma u$. $\square$

Considering the previous theorem, the action of $\mathcal{S}_n$ on the set of λ-tableaux induces a well defined action on the set of λ-tabloids by letting

$$\sigma\{t\} := \{\sigma t\}.$$

Definition 2.2.6. For $\lambda \vdash n$, let $\{t_1, t_2, ..., t_k\}$ be the collection of all λ-tabloids, where $k = \frac{n!}{\lambda_1!\lambda_2!...\lambda_l!}$. Then

$$M^\lambda = \mathbb{C}\{\{t_1\}, \{t_2\}, ..., \{t_k\}\}$$

is the *permutation module associated with* λ.

Let G be a group. A G-module is *cyclic* if it can be generated (as a module) by one element.

Proposition 2.2.7. *Let $\lambda \vdash n$. For any $t \in \mathrm{Tab}(\lambda)$, M^λ is a cyclic $\mathcal{S}_n$-module, generated by $\delta_{\{t\}}$.*

Proof. The action of $\mathcal{S}_n$ is transitive on $\mathrm{Tab}(\lambda)$. In other words, for any pair $t, u \in \mathrm{Tab}(\lambda)$, there exists a permutation σ such that $u = \sigma t$. Consequently the induced action on the set of λ-tabloids is also transitive. Therefore M^λ is a cyclic $\mathcal{S}_n-$module, generated by any $\delta_{\{t\}}$. $\square$

2.3 Specht modules

With the establishment of the permutation modules, each irreducible $\mathcal{S}_n$-module can now be constructed. For a partition λ, the corresponding irreducible will be generated inside of M^λ by a family of elements created from the tableaux associated to λ.

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, and t be a λ-tableau. From t, one will need an element of $\mathbb{C}[\mathcal{S}_n]$ to act as an operator on M^λ. Note that there are λ_1 columns in the shape of λ, and that λ_j^* is length of the jth column in the shape of λ. For each $j \in [\lambda_1]$ and $i \in [l]$, set

$$\lambda_j^* = \max\{k \in [l] \mid \lambda_k \geq j\},$$

and define

$$R_i := \{t_{i,j} \mid j \in [\lambda_i]\} \qquad C_j := \{t_{i,j} \mid i \in [\lambda_j^*]\}.$$

Definition 2.3.1. Let $R_1, R_2, ..., R_l$ be the rows of t, and $C_1, C_2, ..., C_{\lambda_1}$ be the corresponding columns. Then

(1) $R_t := \mathcal{S}_{R_1} \times \mathcal{S}_{R_2} \times ... \times \mathcal{S}_{R_l}$

(2) $C_t := \mathcal{S}_{C_1} \times \mathcal{S}_{C_2} \times ... \times \mathcal{S}_{C_{\lambda_1}}$

are the *row-stabilizer* and the *column-stabilizer* of t, respectively.

Considering Definition 2.3.1, for a subset $H \subseteq \mathcal{S}_n$, set

(1) $H^+ := \sum\limits_{\sigma \in H} \sigma$

(2) $H^- := \sum\limits_{\sigma \in H} \mathrm{sgn}(\sigma)\sigma$,

and define the following element of $\mathbb{C}[\mathcal{S}_n]$,

$$\kappa_t := C_t^-.$$

Definition 2.3.2. Let t be a λ-tableau. The *polytabloid of type* t is the element

$$e_t := \kappa_t \delta_{\{t\}}.$$

Furthermore, the $\mathcal{S}_n$-submodule generated by all the polytabloids,

$$\mathcal{S}^\lambda = \langle \sigma e_t \in M^\lambda | \sigma \in \mathcal{S}_n,\ t \in \mathrm{Tab}(\lambda) \rangle,$$

is the associated *Specht module*.

The claim is that the arrival of Specht modules ends the search for all the irreducible group representations of $\mathcal{S}_n$. With that said, the next lemma will be helpful in providing justification to this claim.

Lemma 2.3.3. *Let t be a tableau, and $\sigma \in \mathcal{S}_n$. Then*

(1) $R_{\sigma t} = \sigma R_t \sigma^{-1}$,

(2) $C_{\sigma t} = \sigma C_t \sigma^{-1}$,

(3) $\kappa_{\sigma t} = \sigma \kappa_t \sigma^{-1}$,

(4) $e_{\sigma t} = \sigma e_t$.

Proof. For part 1, note $\tau \in R_t$ if and only if $\{\tau t\} = \{t\}$. Write $\{\tau t\} = \{\tau \sigma^{-1}(\sigma t)\}$. Then $\{\tau \sigma^{-1}(\sigma t)\} = \{t\}$. Thus

$$\{\sigma \tau \sigma^{-1}(\sigma t)\} = \sigma\{\tau \sigma^{-1} \sigma t\} = \sigma\{t\} = \{\sigma t\},$$

and hence $\sigma \tau \sigma^{-1} \in R_{\sigma t}$. Therefore $\sigma R_t \sigma^{-1} \subseteq R_{\sigma t}$.

Conversely, replace t with σt, and σ with σ^{-1} to conclude $\sigma^{-1} R_{\sigma t} \sigma \subseteq R_t$, and equivalently, $R_{\sigma t} \subseteq \sigma R_t \sigma^{-1}$

The proof of part 2 is analogous to part 1, by considering the dual notion of *column-equivalence* between λ-tableaux.

For part 3, note that, for any $\sigma, \tau \in \mathcal{S}_n$, $\mathrm{sgn}(\sigma \tau \sigma^{-1}) = \mathrm{sgn}(\tau)$. Thus

$$\sigma \kappa_t \sigma^{-1} = \sum_{\tau \in C_t} \mathrm{sgn}(\tau) \sigma \tau \sigma^{-1} = \sum_{\tau \in C_t} \mathrm{sgn}(\sigma \tau \sigma^{-1}) \sigma \tau \sigma^{-1} = \kappa_{\sigma t}.$$

Finally, part 4 follows from the observation,

$$\sigma e_t = \sigma \kappa_t \delta_{\{t\}} = (\sigma \kappa_t \sigma^{-1}) \sigma \delta_{\{t\}} = \kappa_{\sigma t} \delta_{\{\sigma t\}} = e_{\sigma t}.$$

□

An immediate consequence of part (4) of Lemma 2.3.3 is that the Specht module $\mathcal{S}^\lambda$ is generated by any λ-tableau t, i.e. $\mathcal{S}^\lambda = \langle \sigma e_t | \sigma \in \mathcal{S}_n \rangle$. In addition,

$$\mathcal{S}^\lambda = \mathrm{span}\{e_t \mid t \in \mathrm{Tab}(\lambda)\},$$

and therefore $\mathcal{S}^\lambda$ has a basis consisting of some collection of polytabloids $\{e_{t_i} \mid i \in [\dim \mathcal{S}^\lambda]\}$.

2.4 Orderings on shapes

The following partial order will be needed in showing that the Specht modules arising from two different partitions of n are inequivalent.

Definition 2.4.1. Suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$ and $\mu = (\mu_1, \mu_2, ..., \mu_k) \vdash n$. Then λ *dominates* μ ,written $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + ... + \lambda_i \geq \mu_1 + \mu_2 + ... + \mu_i,$$

for all $i \geq 1$. If $i > l$ (respectively $i > k$), then λ_i (respectively μ_i) is taken to be zero.

Theorem 2.4.2. *The relation $\trianglerighteq$ is a partial order on the set of partitions of n.*

Proof. The relation $\trianglerighteq$ is reflexive and transitive since the standard order $\geq$ on $\mathbb{Z}$ is reflexive and transitive.

Now suppose $\lambda \trianglerighteq \mu$ and $\mu \trianglerighteq \lambda$ for $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$ and $\mu = (\mu_1, \mu_2, ..., \mu_k) \vdash n$. Then $\lambda_1 + \lambda_2 + ... + \lambda_i = \mu_1 + \mu_2 + ... + \mu_i$, for each $i \geq 1$. Thus $\lambda_1 = \mu_1$. Using induction, one verifies $\lambda_i = \mu_i$ for each $i \in [l] = [k]$. Hence $\lambda = \mu$, and therefore $\trianglerighteq$ is anti-symmetric. □

The utility of this partial ordering will be from the application of the following lemma.

Lemma 2.4.3. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, $\mu = (\mu_1, \mu_2, ..., \mu_k) \vdash n$, and let t and u be tableaux of shapes λ and μ, respectively. If, for all $i \in [k]$, the entries of the ith row of u appear in different columns of t, then $\lambda \trianglerighteq \mu$.*

Proof. First note that if $\sigma \in \mathcal{S}_n$ preserves the columns of t, then the hypothesis still holds for u and σt.

Now, by hypothesis, one can first permute the entries of each column of t so that the elements of $\{u_{1,j} \mid j \in [\mu_1]\}$ all appear in the first row of t_1, where t_1 is the new tableau obtained from the permutation. Thus $\lambda_1 \geq \mu_1$.

Again, the hypothesis holds for u and t_1. So, permute the entries of each column of t_1 not belonging to the first row of u so that each elements of $\{u_{2,j} \mid j \in [\mu_2]\}$ also appears in the first and second row of t_2, where t_2 is the new tableau obtained from the permutation. (To be exact, in t_1, the entry $u_{2,j}$ will go to the first position of its residing column, or to the second position of that column if the first position is already taken by an entry from the first row of u.) Hence $\lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$.

Finally, one repeats this procedure up through the ith row of u so that the corresponding entries appear in the first i rows of some tableau of shape λ. Thus

$$\lambda_1 + \lambda_2 + ... + \lambda_i \geq \mu_1 + \mu_2 + ... + \mu_i,$$

and therefore $\lambda \trianglerighteq \mu$. □

This section concludes with another useful result. It doesn't pertain to the partial order $\trianglerighteq$, but its placement is justified due to its similarity with the previous lemma.

Lemma 2.4.4. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, and t, u be λ-tableaux. If, for each $i \in [\lambda_1]$, the entries of the ith column of t appear in different rows of u, then for some $\sigma \in C_t$,*

$$\{u\} = \{\sigma t\}.$$

Proof. Like the proof of Lemma 2.4.3, by hypothesis, one can permute the entries in each row of u to obtain a new tableau u_1 such that, for each i, the ith column of u_1 and t consist of the same entries. In other words, for some $\sigma \in C_t$,

$$u_1 = \sigma t.$$

Therefore $\{u\} = \{\sigma t\}$ since u_1 and u are row equivalent. □

2.5 The submodule theorem

It will now be shown that the collection of all Specht modules $\mathcal{S}^\lambda$ forms a complete set of distinct irreducible $\mathcal{S}_n$-modules. Using the results established in Section 1.5, the number of distinct irreducible $\mathcal{S}_n$-modules equals the number of partitions of n. Therefore the objective is to verify that each Specht module is irreducible, and that different partitions of n yield inequivalent modules. Note that the bulk of this section is comprised of preliminary results needed for this argument.

Lemma 2.5.1. *Let $H \leq \mathcal{S}_n$, and $\sigma \in \mathcal{S}_n$. If H contains an odd permutation, then half of the permutations in σH are odd and half of the permutations are even.*

Proof. First, for any $\sigma \in \mathcal{S}_n$, $\operatorname{sgn}(\sigma) = 1$ if and only if σ is even. Suppose now $\sigma \in H$ is odd. Then

$$\operatorname{sgn}|_H \to \{\pm 1\}$$

is a surjective group homomorphism. Therefore the result follows by the first isomorphism theorem with Lagrange's theorem for finite groups, i.e.

$$|H| = [H : \ker(\operatorname{sgn}|_H)] \cdot |\ker(\operatorname{sgn}|_H)| = 2 \cdot |\ker(\operatorname{sgn}|_H)|.$$

□

Lemma 2.5.2. *Suppose $H \leq \mathcal{S}_n$ acts on a set X, and let $x \in X$. If H_x, the stabilizer of x, contains an odd permutation, then*

$$H^- \delta_x = 0.$$

Proof. Let $\sigma, \tau \in H$. Note that if $\tau \in \sigma H_x$, then $\tau \delta_x = \sigma \delta_x$. Thus

$$\sum_{\tau \in \sigma H_x} \operatorname{sgn}(\tau) \tau \delta_x = \sum_{\tau \in \sigma H_x} \operatorname{sgn}(\tau) \sigma \delta_x = s \sigma \delta_x$$

where $s = \sum_{\tau \in \sigma H_x} \operatorname{sgn}(\tau)$. But by Lemma 2.5.1, σH_x contains an equal number of even and odd permutations, since H_x contains an odd permutation. Thus $s = 0$, and hence

$$\sum_{\tau \in \sigma H_x} \operatorname{sgn}(\tau)\tau\delta_x = 0.$$

Finally, let $\{\sigma_1, \sigma_2, ..., \sigma_k\}$ be a transversal for H_x in H. Then

$$\sum_{\tau \in H} \operatorname{sgn}(\tau)\tau\delta_x = \sum_{i=1}^{k} \sum_{\tau \in \sigma_i H_x} \operatorname{sgn}(\tau)\tau\delta_x.$$

Therefore $H^- \delta_x = 0$. □

Proposition 2.5.3. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ and $\mu = (\mu_1, \mu_2, ..., \mu_k)$ be partitions of n, and let t and u be tableaux of shape λ and μ, respectively. If $\lambda \not\trianglerighteq \mu$, then*

$$\kappa_t \delta_{\{u\}} = 0.$$

Proof. First, the contraposition of the implication in Lemma 2.4.3 states that if $\lambda \not\trianglerighteq \mu$, then, for each $t \in \operatorname{Tab}(\lambda)$ and $u \in \operatorname{Tab}(\mu)$, there is some row of u containing two entries i and j, which both lie in some same column of t. Thus the stabilizer of $\{u\}$ in C_t contains the transposition (i, j). Therefore by Lemma 2.5.2,

$$\kappa_t \delta_{\{u\}} = C_t^- \delta_{\{u\}} = 0.$$

□

Note that an equivalent statement of this proposition is that if $\kappa_t \neq 0$ as a linear operator on M^μ, then $\lambda \trianglerighteq \mu$.

Lemma 2.5.4. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash n$, and t, u be tableaux of shape λ. Then*

$$\kappa_t \delta_{\{u\}} = \begin{cases} \operatorname{sgn}(\sigma) e_t & \text{if } \{u\} = \{\sigma t\} \text{ for some } \sigma \in C_t \\ 0 & \text{if } \{u\} \neq \{\sigma t\} \text{ for every } \sigma \in C_t \end{cases}.$$

In particular, $\kappa_t f$ is a scalar multiple of e_t for all $f \in M^\lambda$.

Proof. Suppose that $\{u\} = \{\sigma t\}$ for $\sigma \in C_t$. Then

$$\kappa_t \delta_{\{u\}} = \sum_{\tau \in C_t} \operatorname{sgn}(\tau)\tau\delta_{\{\sigma t\}} = \sum_{\tau \in C_t} \operatorname{sgn}(\tau)(\tau\sigma)\delta_{\{t\}} = \operatorname{sgn}(\sigma) e_t$$

since $\operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau\sigma)$.

Now suppose $\{u\} \neq \{\sigma t\}$ for any $\sigma \in C_t$. Then by Lemma 2.4.4, there is some column of t containing two entries i and j, which both lie in some same row of u. Thus the stabilizer of $\{u\}$ in C_t contains the transposition (i, j). Therefore by Lemma 2.5.2,

$$\kappa_t \delta_{\{u\}} = C_t^- \delta_{\{u\}} = 0.$$

□

Recall from Section 1.5, M^λ has an invariant inner product $\langle \cdot \mid \cdot \rangle$.

Lemma 2.5.5. *Let H be a subgroup of $\mathcal{S}_n$, and $f, g \in M^\lambda$. Then*

$$\langle f \mid H^- g\rangle = \langle H^- f \mid g\rangle,$$

In other words, H^- is self-adjoint.

Proof. It will be enough to show that the lemma holds for the standard basis. Indeed, let t and u be tableaux of shape λ. Then by Proposition 1.5.2, for any $\sigma \in \mathcal{S}_n$,

$$\langle \delta_{\{t\}} \mid \sigma\delta_{\{u\}}\rangle = \langle \sigma^{-1}\delta_{\{t\}} \mid \delta_{\{u\}}\rangle.$$

Thus

$$\langle \delta_{\{t\}} \mid \operatorname{sgn}(\sigma)\sigma\delta_{\{u\}}\rangle = \langle \operatorname{sgn}(\sigma^{-1})\sigma^{-1}\delta_{\{t\}} \mid \delta_{\{u\}}\rangle$$

since $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$.

Now notice that, since H is a subgroup,

$$\sum_{\sigma\in H} \operatorname{sgn}(\sigma^{-1})\sigma^{-1} = \sum_{\sigma\in H} \operatorname{sgn}(\sigma)\sigma = H^-.$$

Thus

$$\sum_{\sigma\in H} \langle \delta_{\{t\}} \mid \operatorname{sgn}(\sigma)\sigma\delta_{\{u\}}\rangle = \sum_{\sigma\in H} \langle \operatorname{sgn}(\sigma^{-1})\sigma^{-1}\delta_{\{t\}} \mid \delta_{\{u\}}\rangle,$$

and therefore $\langle \delta_{\{t\}} \mid H^-\delta_{\{u\}}\rangle = \langle H^-\delta_{\{t\}} \mid \delta_{\{u\}}\rangle$. □

Theorem 2.5.6 (The Submodule Theorem)**.** *Suppose W is an $\mathcal{S}_n$−submodule of M^λ. Then $\mathcal{S}^\lambda \subseteq W$, or $\mathcal{S}^\lambda \perp W$. In particular, each $\mathcal{S}^\lambda$ is irreducible.*

Proof. Let $v \in W$, $f \in \mathcal{S}^\lambda$, and $u \in \operatorname{Tab}(\lambda)$. Suppose that, for every $w \in W$,

$$\langle w \mid e_u\rangle = 0.$$

First, since $\mathcal{S}^\lambda$ is generated by any tableau of shape λ, for some collection of complex numbers $\{c_\sigma\}$,

$$f = \sum_{\sigma\in\mathcal{S}_n} c_\sigma \sigma e_u.$$

Set $F = \sum\limits_{\sigma\in\mathcal{S}_n} c^*_\sigma \sigma^{-1} \in \mathbb{C}[\mathcal{S}_n]$, where c^*_σ is complex conjugate of c_σ, and note that, by Proposition 1.5.2,

$$\langle v \mid f\rangle = \langle Fv \mid e_u\rangle.$$

However, W is a submodule. Thus $Fv \in W$. But by the assumption, $\langle Fv \mid e_u\rangle = 0$. Therefore $\langle v \mid f\rangle = 0$, and hence $\mathcal{S}^\lambda \subseteq W^\perp$.

Now suppose $\mathcal{S}^\lambda \nsubseteq W^\perp$. Then for $u \in \operatorname{Tab}(\lambda)$, there is some $w \in W$, such that

$$\langle w \mid e_u\rangle \neq 0.$$

By Lemma 2.5.5, $\langle w \mid e_u \rangle = \langle \kappa_u w \mid \delta_{\{u\}} \rangle$. Thus $\kappa_u w \neq 0$. But $\kappa_u w = a e_u$ for some $a \in \mathbb{C}$. Hence, $e_u \in W$ since W is invariant. Therefore $\mathcal{S}^\lambda \subseteq W$ since $\mathcal{S}^\lambda$ is generated by e_u, and W is a submodule.

To see that $\mathcal{S}^\lambda$ is irreducible. Apply Maschke's Theorem to M^λ, and note that $\mathcal{S}^\lambda$ cannot be orthogonal to every irreducible submodule in the decomposition of M^λ. □

Proposition 2.5.7. *Let $\lambda, \mu \vdash n$, and suppose there is some nonzero $\phi \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}^\lambda, M^\mu)$. Then $\lambda \trianglerighteq \mu$, and if $\lambda = \mu$, then ϕ is a scalar.*

Proof. Suppose $\phi \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}^\lambda, M^\mu)$ is nonzero. Then there is some $t \in \mathrm{Tab}(\lambda)$, such that $\phi(e_t) \neq 0$, for basis vector e_t. Note that

$$\kappa_t \phi(e_t) = \phi(\kappa_t e_t),$$

since ϕ is a homomorphism of $\mathcal{S}_n$-modules. Now

$$\kappa_t e_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma) \sigma e_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma) e_{\sigma t}.$$

Also, $e_{\sigma t} = \mathrm{sgn}(\sigma) e_t$ whenever $\sigma \in C_t$. Thus $\kappa_t e_t = \mid C_t \mid e_t$, and consequently

$$\kappa_t \phi(e_t) = \mid C_t \mid \phi(e_t) \neq 0.$$

So κ_t is nonzero as a linear operator on M^μ. Therefore by Proposition 5.3.5, $\lambda \trianglerighteq \mu$.

In addition, suppose $\lambda = \mu$. Then by Lemma 2.5.4, $\kappa_t \phi(e_t) = c e_t$ for some nonzero $c \in \mathbb{C}$. ($c \neq 0$, since $\kappa_t \phi(e_t) \neq 0$) But it has just been shown that $\kappa_t \phi(e_t) = \mid C_t \mid \phi(e_t)$. Thus

$$\phi(e_t) = \frac{c}{\mid C_t \mid} e_t.$$

Finally, suppose e_u is another basis vector of $\mathcal{S}^\lambda$ for $u \in \mathrm{Tab}(\lambda)$, and let $\sigma \in \mathcal{S}_n$ such that $u = \sigma t$. Then by Lemma 2.5.4, $e_u = \sigma e_t$. Thus

$$\phi(e_u) = \phi(\sigma e_t) = \sigma \phi(e_t) = \sigma(\frac{c}{\mid C_t \mid} e_t) = \frac{c}{\mid C_t \mid} \sigma e_t = \frac{c}{\mid C_t \mid} e_u.$$

Therefore ϕ is a scalar. □

Theorem 2.5.8. *The collection of all Specht modules*

$$\{\mathcal{S}^\lambda \mid \lambda \vdash n\}$$

forms a complete set of irreducible $\mathcal{S}_n-$modules.

Proof. First, by Theorem 2.5.6, all the Specht modules are irreducible. All that remains then is verifying that, for any $\lambda, \mu \vdash n$,

$$\mathcal{S}^\lambda \cong \mathcal{S}^\mu$$

if and only if $\lambda = \mu$.

Suppose $\mathcal{S}^\lambda \cong \mathcal{S}^\mu$. Then by definition, there exists some invertible $\phi \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}^\lambda, \mathcal{S}^\mu)$. But, $\mathcal{S}^\mu \subseteq M^\mu$. Thus $\phi \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}^\lambda, M^\mu)$. Therefore $\lambda \trianglerighteq \mu$ by Proposition 2.5.7. On the other hand, $\phi^{-1} \in \mathrm{Hom}_{\mathcal{S}_n}(\mathcal{S}^\mu, \mathcal{S}^\lambda)$. Thus $\mu \trianglerighteq \lambda$ as well. Therefore $\lambda = \mu$.

Finally, the set of distinct irreducible $\mathcal{S}_n-$modules is in one to one correspondence with the set of partitions of n. Therefore $\{\mathcal{S}^\lambda \mid \lambda \vdash n\}$ gives the full set of distinct irreducible $\mathcal{S}_n-$modules. □

This section closes with a convenient computational rule to calculate the the dimension of each Specht module of a given integer $n \geq 1$. Unfortunately, the proof of the validity of the formula will be omitted as justification is far from trivial. See Appendix C in [4] for proof. Let $\lambda \vdash n$. The *hook length* of a position in the Ferrer's diagram of λ is the number the positions to its right plus the number of positions below it plus one.

Theorem 2.5.9. *Let $\lambda \vdash n$, and define h_λ to be the product of all hook lengths in λ. Then*

$$\dim \mathcal{S}^\lambda = \frac{n!}{h_\lambda}.$$

2.6 General projection operators

To get full use out of $\mathcal{S}_n$, a more tangible collection of irreducible modules will be utilized. To explain, in the next chapter it will be shown that the modules carrying the irreducible tensor product representation of $\mathrm{GL}(m, \mathbb{C})$ can be constructed as the images of specific operators built from the integer partitions. It is the purpose of this section to define these elements, and through the aid of the Specht modules, obtain these new versions of irreducible $\mathcal{S}_n$-modules.

Let t be a tableau of shape λ. For the following definition, set

$$\iota_t := R_t^+$$

Definition 2.6.1. Let t be a tableau of shape $\lambda \vdash n$. The *general projection operator of type* t is

$$\epsilon_t := \kappa_t \iota_t$$

Note that ϵ_t is not a true *projection* since in general, $\epsilon_t^2 \neq \epsilon_t$ but rather, $\epsilon_t^2 = C\epsilon_t$ for some constant C.

Example 2.6.2. Recall Example 2.1.2. Let t be the tableau of shape $\lambda_1 = (4) \vdash 4$ defined by

$$t_{1,j} := j$$

for $j \in [4]$. Then $\epsilon_t = \sum_{\sigma \in \mathcal{S}_4} \sigma$. Note that, ϵ_{t_1} only differs by a constant from the projection operator of the subspace of symmetric tensors inside the 4th-fold tensor power of some arbitrary finite-dimensional vectors space. Likewise, now let t be the tableau of shape $\lambda_3 = (1, 1, 1, 1) \vdash 4$ defined by

$$t_{i,1} := i$$

for $i \in [4]$. Here, $\epsilon_t = \sum\limits_{\sigma \in \mathcal{S}_4} \operatorname{sgn} \sigma$, which itself only differs by constant from the projection operator of the subspace of anti-symmetric rank-4 tensors. Finally, let t be the tableau of shape $\lambda_2 = (2,2) \vdash 4$ given by

$$\begin{aligned} t_{1,1} &= 1 \qquad & t_{1,2} &= 2 \\ t_{2,1} &= 3 \qquad & t_{2,2} &= 4. \end{aligned}$$

Then $R_t = \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{3,4\}}$, $C_t = \mathcal{S}_{\{1,3\}} \times \mathcal{S}_{\{2,4\}}$, and

$$\epsilon_t = ((\,) - (1\,3) - (2\,4) + (1\,3)(2\,4))\,((\,) + (1\,2) + (3\,4) + (1\,2)(3\,4))\,,$$

where for simplicity, $(\,)$ denotes the identity permutation. After some calculation,

$$\epsilon_t = (\,) + (1\,2) - (1\,3) - (2\,4) + (3\,4) - (1\,2\,3) - (1\,3\,4) - (1\,4\,2)...$$

$$... - (2\,4\,3) + (1\,4)(2\,3) + (1\,3\,2\,4) - (1\,3\,4\,2) + (1\,4\,2\,3) - (1\,4\,3\,2).$$

It too will define some subspace of rank-4 tensors. Thus this exercise shows the motivation for the name *general projection operator*.

Definition 2.6.3. Let R be a ring and let $a \in R$. Then the *principal left ideal generated by* a is

$$\langle a \rangle_L := \{ra \mid r \in R\}$$

Theorem 2.6.4. *Let t be a tableau of shape $\lambda \vdash n$. Then*

$$\langle \iota_t \rangle_L \cong M^\lambda \quad \textit{and} \quad \langle \epsilon_t \rangle_L \cong \mathcal{S}^\lambda$$

as $\mathcal{S}_n$−modules. In particular, $\langle \epsilon_t \rangle_L$ is irreducible.

Proof. First, M^λ is generated by $\delta_{\{t\}}$. So let $\phi : \langle \iota_t \rangle_L \to M^\lambda$ be the linear extension of the following assignment:

$$\phi(\sigma \iota_t) \mapsto \delta_{\{\sigma t\}},$$

for each $\sigma \in \mathcal{S}_n$. In other words, if $f = \sum\limits_{\sigma \in \mathcal{S}_n} a_\sigma \sigma \in \mathbb{C}[\mathcal{S}_n]$, then

$$\phi(f \iota_t) = \sum_{\sigma \in \mathcal{S}_n} a_\sigma \delta_{\{\sigma t\}}.$$

To verify that ϕ is well-defined, let $\sigma, \tau \in \mathcal{S}_n$, and suppose $\sigma \iota_t = \tau \iota_t$. Then $(\tau^{-1}\sigma)\iota_t = \iota_t$, and

$$\sum_{\pi \in R_t} (\tau^{-1}\sigma)\pi = \sum_{\pi \in R_t} \pi$$

if and only if $(\tau^{-1}\sigma) \in R_t$. Consequently, the tableau $\tau^{-1}\sigma t$ is row equivalent to t. Hence $\{\tau^{-1}\sigma t\} = \{t\}$, and $\delta_{\{\tau^{-1}\sigma t\}} = \delta_{\{t\}}$. Thus

$$\tau^{-1}\sigma \delta_{\{t\}} = \delta_{\{t\}},$$

and $\sigma\delta_{\{t\}} = \tau\delta_{\{t\}}$. Therefore $\phi(\sigma\iota_t) = \phi(\tau\iota_t)$.

Furthermore, ϕ was constructed to be a $\mathcal{S}_n-$homomorphism. To see this, let $\tau \in \mathcal{S}_n$. Then

$$\tau f = \sum_{\sigma \in \mathcal{S}_m} a_\sigma \tau\sigma$$

for $f = \sum\limits_{\sigma \in \mathcal{S}_m} a_\sigma \sigma \in \mathbb{C}[\mathcal{S}_m]$. Thus

$$\begin{aligned}
\phi(\tau(f\iota_t)) &= \phi\left((\tau f)(\iota_t)\right) \\
&= \sum_{\sigma \in \mathcal{S}_m} a_\sigma \delta_{\{\tau\sigma t\}} \\
&= \sum_{\sigma \in \mathcal{S}_m} a_\sigma \tau\delta_{\{\sigma t\}} \\
&= \tau(\sum_{\sigma \in \mathcal{S}_m} a_\sigma \delta_{\{\sigma t\}}) \\
&= \tau(\phi(f\iota_t)).
\end{aligned}$$

Finally, ϕ is a bijection. Indeed, let $d \in M^\lambda$. Now M^λ is cyclic. Thus $d = \sum\limits_{\sigma \in \mathcal{S}_n} a_\sigma \sigma\delta_{\{t\}}$, for some collection of constants $\{a_\sigma\}_\sigma \subseteq \mathbb{C}$. With this, if $f = \sum\limits_{\sigma \in \mathcal{S}_m} a_\sigma\sigma \in \mathbb{C}[\mathcal{S}_m]$, then

$$\phi(f\iota_t) = \sum_{\sigma \in \mathcal{S}_m} a_\sigma \delta_{\{\sigma t\}} = \sum_{\sigma \in \mathcal{S}_m} a_\sigma \sigma\delta_{\{t\}} = d.$$

Therefore ϕ is onto. Now, suppose $f\iota_t \in \ker\phi$. Then

$$\begin{aligned}
0 &= \phi(f\iota_t) \\
&= \sum_{\sigma \in \mathcal{S}_n} a_\sigma \delta_{\{\sigma t\}}.
\end{aligned}$$

But, $\{\delta_{\{\sigma t\}} \mid \sigma \in \mathcal{S}_n\}$ is a basis for M^λ. Thus $a_\sigma = 0$ for all $\sigma \in \mathcal{S}_n$. Therefore $f\iota_t = 0$, and

$$\langle \iota_t \rangle_L \cong M^\lambda$$

as $\mathcal{S}_n-$ modules.

For the second assertion, note that $\langle \epsilon_t \rangle_L \leq \langle \iota_t \rangle_L$ since $\epsilon_t = \kappa_t \iota_t$. Furthermore,

$$\phi(\langle \epsilon_t \rangle_L) = \mathcal{S}^\lambda.$$

Indeed, $\mathcal{S}^\lambda = \langle \sigma e_t \mid \sigma \in \mathcal{S}_n \rangle$ and

$$\begin{aligned} \phi(\epsilon_t) &= \phi\left(\kappa_t(\iota_t)\right) \\ &= \kappa_t(\phi(\iota_t)) \\ &= \kappa_t \delta_{\{t\}} \\ &= e_t. \end{aligned}$$

Thus $\phi(\epsilon_t)$ generates $\mathcal{S}^\lambda$ since ϕ is a $\mathcal{S}_n$-homomorphism. Therefore $\langle \epsilon_t \rangle_L \cong \mathcal{S}^\lambda$. $\square$

Chapter 3

Irreducible tensor representations of $\mathrm{GL}(n, \mathbb{C})$

As an infinite group, the representation theory for $\mathrm{GL}(n, \mathbb{C})$ is more complicated than the theory for finite groups. For example, Maschke's Theorem and the classification methods established in the Chapter 1 cannot be applied. Fortunately, the symmetric group can be utilized to find the irreducible representations for $\mathrm{GL}(n, \mathbb{C})$, carried in the various tensor powers of $\mathbb{C}^n$. Yes, a simple connection between $\mathcal{S}_m$ and $\mathrm{GL}(n, \mathbb{C})$ allows for the association of the irreducible tensor representations with integer partitions, along with the obtainment of the corresponding irreducible $\mathrm{GL}(n, \mathbb{C})$-modules as image spaces of the general projection operators introduced in Section 2.6. Ultimately, these irreducible $\mathrm{GL}(n, \mathbb{C})$-modules are the source for all the irreducible representations of the special unitary group, $\mathrm{SU}(n)$.

The results needed from this chapter will be established in complete generality using the general linear group of an arbitrary finite dimensional complex vector space, denoted as V, with $n := \dim V$. These methods are modeled by the treatment offered by Sternberg [4].

For a non-negative integer m, let

$$V^{\bigotimes m} := \underbrace{V \otimes V \otimes ... \otimes V}_{m-\text{times}}$$

denote the $m-$fold tensor product space of V. By convention, $V^{\bigotimes 0} := \mathbb{C}$. However, this case is of no interest here. Thus one assumes that $m \geq 1$.

Note that tensors of the form $v_1 \otimes v_2 \otimes ... \otimes v_m$ are called *monomials*, and

$$V^{\bigotimes m} = \operatorname{span}\{v_1 \otimes v_2 \otimes ... \otimes v_m \mid v_i \in V \; i \in [m]\}.$$

In fact, one has the following.

Proposition 3.0.1. *Suppose $\mathcal{B} = \{e_1, e_2, ..., e_m\}$ is a basis for V. Then*

$$\mathcal{B}^m = \{e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m} \mid (i_1, i_2, ..., i_m) \in [n]^m\}$$

is a basis for $V^{\bigotimes m}$.

Sometimes for convenience, $e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}$ will be denoted as $\mathbf{e}_I$, where $I = (i_1, i_2, ..., i_m)$.

3.1 $V^{\bigotimes m}$ as a $\mathrm{GL}(V)$-module and $\mathcal{S}_m-$module.

The symmetric group acts on the set of monomials by setting

$$\sigma(v_1 \otimes v_2 \otimes ... \otimes v_m) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes ... \otimes v_{\sigma^{-1}(m)}$$

for all $\sigma \in \mathcal{S}_m$. This is a group action. One can consider the monomial $v_1 \otimes v_2 \otimes ... \otimes v_m$ as an assignment from $[m]$ to V defined by

$$i \mapsto v_i.$$

So, if $i \mapsto v_i$, then the vector $v_i \in V$ is in the ith position of the monomial. Therefore $\mathcal{S}_m$ acts on monomials as it would on functions from $[m]$ to V, i.e. $\sigma f = f \circ \sigma^{-1}$. The m-fold tensor product space $V^{\bigotimes m}$ becomes an $\mathcal{S}_m$-module when this action extends by linearity from monomials to all tensors.

In addition to this, $\mathrm{GL}(V)$ also acts on monomials by setting

$$A^{\otimes m}(v_1 \otimes v_2 \otimes ... \otimes v_m) := A(v_1) \otimes A(v_2) \otimes ... \otimes A(v_m),$$

for all $A \in \mathrm{GL}(V)$. This is also a group action since

$$A^{\otimes m} B = A^{\otimes m} B^{\otimes m}.$$

The space, $V^{\bigotimes m}$ becomes a $\mathrm{GL}(V)$-module when this action extends by linearity from monomials to all tensors. From this point on, the $\mathrm{GL}(n, \mathbb{C})$-representation will be denoted as $T^{\otimes m}$, i.e.

$$T^{\otimes m} : A \mapsto A^{\otimes m}.$$

3.2 The commuting action of $\mathrm{GL}(V)$ and $\mathcal{S}_m$ on $V^{\bigotimes m}$.

Considering the defining actions of $\mathrm{GL}(V)$ and $\mathcal{S}_m$ on $V^{\bigotimes m}$, one has

$$\sigma \circ A^{\otimes m} = A^{\otimes m} \circ \sigma,$$

for all $\sigma \in \mathcal{S}_m$ and $A \in \mathrm{GL}(V)$. What will need to be shown is that

$$\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \mathrm{span}\{A^{\otimes m} \mid A \in \mathrm{GL}(V)\}.$$

Ultimately, the ability to obtain the desired irreducible representations of $\mathrm{GL}(V)$ rests on this result. Therefore the remainder of this section is set to justify this identity.

By fixing a basis $\mathcal{B} = \{w_i \mid i \in [n]\}$ on V one can identify $\mathrm{Hom}_{\mathbb{C}}(V, V)$ with $\mathbb{C}^{n^2}$, endowing $\mathrm{Hom}_{\mathbb{C}}(V, V)$ with the metric space induced from the euclidean topology.

Lemma 3.2.1. *Let V be a vector space. Then for any $A \in \mathrm{Hom}_{\mathbb{C}}(V, V)$, there exists a* *sequence* $(B_n)_{n\in\mathbb{N}}$ *in* $\mathrm{GL}(V)$*, such that*

$$\lim_{n\to\infty} B_n = A.$$

In particular,

$$\mathrm{Hom}_{\mathbb{C}}(V,V) = \overline{\mathrm{GL}(V)}.$$

Proof. It is sufficient to prove this using matrices. Let $A \in M_n(\mathbb{C}) \setminus \mathrm{GL}(n,\mathbb{C})$. Since $\mathbb{C}$ is algebraically closed, A is similar, via some $P \in \mathrm{GL}(n,\mathbb{C})$, to the block diagonal matrix

$$C = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{bmatrix}$$

for some $l \geq 1$, where each J_i, is a Jordan block corresponding to an eigenvalue of A. So, for each $k \geq 1$, define $C(k)$ by replacing every eigenvalue equal to zero with the number $\frac{1}{k}$. Now, set $B_k = P^{-1}C(k)P$. Then

$$\lim_{k\to\infty} B_k = P^{-1}CP = A.$$

Therefore $M_n(\mathbb{C}) = \overline{\mathrm{GL}(n,\mathbb{C})}$. □

Lemma 3.2.2. *For each* $m \geq 1$*, the assignment*

$$\varphi : A_1 \otimes A_2 \otimes ... \otimes A_m \mapsto \varphi(A_1 \otimes A_2 \otimes ... \otimes A_m)$$

defined by setting

$$\varphi(A_1 \otimes A_2 \otimes ... \otimes A_m)(v_1 \otimes v_2 \otimes ... \otimes v_m) := A_1(v_1) \otimes A_2(v_2) \otimes ... \otimes A_m(v_m),$$

for monomials, extends by linearity to an isomorphism

$$\varphi : \mathrm{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V,V)^{\bigotimes m}.$$

Proof. Since

$$\dim \mathrm{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m}) = n^{2m} = \dim\left(\mathrm{Hom}_{\mathbb{C}}(V,V)^{\bigotimes m}\right),$$

the goal is to verify that the linear extension of φ (still denoted as φ) is a surjection.

Let

$$\mathcal{B}^m = \{e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m} \mid I = (i_1, i_2, ..., i_m) \in [n]^m\}$$

be a basis for $V^{\bigotimes m}$, induced by a basis $\mathcal{B} = \{e_1, e_2, ..., e_m\}$ of V. Let

$$B \in \mathrm{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m}).$$

Then for each $I =\in [n]^m$,

$$B(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}) = \sum_{J\in[n]^m} B_{JI}\mathbf{e}_J.$$

Consider

$$A_1 \otimes A_2 \otimes ... \otimes A_m \in \mathrm{Hom}_{\mathbb{C}}(V,V)^{\bigotimes m}.$$

If $[A_k]_{\mathcal{B}}$ has entries $a(k)_{j_k i_k}$, then

$$\begin{aligned}\varphi(A_1 \otimes A_2 \otimes ... \otimes A_m)(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}) &= A_1(e_{i_1}) \otimes A_2(e_{i_2}) \otimes .... \otimes A_m(e_{i_m}) \\ &= \sum_{J\in[n]^m} a(1)_{j_1 i_1} a(2)_{j_2 i_2} ... a(m)_{j_m i_m} \mathbf{e}_J.\end{aligned}$$

Now, it is possible to define $A_1 \otimes A_2 \otimes ... \otimes A_m$ so that

$$a(1)_{j_1 i_1} a(2)_{j_2 i_2} ... a(m)_{j_m i_m} := B_{JI}$$

for all $I, J \in [n]^m$. Therefore φ is surjective. □

For the following, the abuse of notation

$$v \otimes v \otimes ... \otimes v \equiv \underbrace{v \otimes v \otimes ... \otimes v}_{m-\text{times}}$$

will be implemented for simplicity. A tensor $\mathbf{z} \in V^{\bigotimes m}$ is *symmetric* if $\sigma\mathbf{z} = \mathbf{z}$ for all $\sigma \in \mathcal{S}_m$.

Lemma 3.2.3. *Let $S^m(V)$ denote the subspace of symmetric tensors in $V^{\bigotimes m}$. Then*

$$S^m(V) = \mathrm{span}\{v \otimes v \otimes ... \otimes v \mid v \in V\}.$$

Proof. The method of proof is mirrored from Sternberg [4], Chapter 5. Let

$$\mathcal{B}^m = \{e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m} \mid I = (i_1, i_2, ..., i_m) \in [n]^m\}$$

be the basis for $V^{\bigotimes m}$, induced by a basis $\mathcal{B} = \{e_1, e_2, ..., e_m\}$ of V. Consider

$$\iota_m \in \mathrm{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m})$$

given by

$$\iota_m := \frac{1}{m!} \sum_{\sigma\in\mathcal{S}_m} \sigma$$

Note that ι_m is a projection operator from $V^{\bigotimes m}$ onto $S^m(V)$. Indeed, let $\mathbf{z} \in S^m(V)$. Then

$\iota_m(\mathbf{z}) = \mathbf{z}$. Conversely, if $\iota_m(\mathbf{z}) = \mathbf{z}$, then, for each $\sigma \in \mathcal{S}_m$,

$$\begin{aligned} \sigma\mathbf{z} &= \sigma(\iota_m(\mathbf{z})) \\ &= (\sigma\iota_m)(\mathbf{z}) \\ &= \iota_m(\mathbf{z}) \\ &= \mathbf{z}. \end{aligned}$$

Now, the collection

$$\{\iota_m(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}) \mid I \in [n]^m\}$$

is a spanning set of $S^m(V)$. Consequently

$$\{\iota_m(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}) \mid 1 \leq i_1 \leq i_2 \leq ... \leq i_m \leq n\}.$$

is a basis for $S^m(V)$.

Consider the monomial $v \otimes v \otimes ... \otimes v$, where $v = \sum_{i_k=1}^n c_k e_k$. First,

$$v \otimes v \otimes ... \otimes v = \sum_{I \in [n]^m} c_{i_i} c_{i_2} ... c_{i_m} e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}.$$

By applying the operator ι_m, one has

$$\iota_m(v \otimes v \otimes ... \otimes v) = \sum_{I \in [n]^m} c_{i_i} c_{i_2} ... c_{i_m} \iota_m(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}). \tag{3.2.1}$$

However,

$$\iota_m(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m}) = \iota_m(\sigma(e_{i_1} \otimes e_{i_2} \otimes .... \otimes e_{i_m})),$$

for all $\sigma \in \mathcal{S}_m$ and $I \in [n]^m$. Consider now

$$\{J \in [n]^m \mid 1 \leq j_1 \leq j_2 \leq ... \leq j_m \leq n\},$$

and set $k(J)_i$ to be the number of times $i \in [n]$ appears in the $m-$tuple J. Using this, the right hand side of 3.2.1 reduces to

$$\sum_J \left(\frac{m!}{k(J)_1! k(J)_2! ... k(J)_n!} c_1^{k(J)_1} c_2^{k(J)_2} ... c_n^{k(J)_n} \right) \iota_m(\mathbf{e}_J).$$

But, $\iota_m(v \otimes v \otimes ... \otimes v) = m! v \otimes v \otimes ... \otimes v$. Therefore

$$v \otimes v \otimes ... \otimes v = \sum_J \left(\frac{c_1^{k(J)_1} c_2^{k(J)_2} ... c_n^{k(J)_n}}{k(J)_1! k(J)_2! ... k(J)_n!} \right) \iota_m(\mathbf{e}_J).$$

Now, consider the coefficients c_i, $i \in [n]$ to be complex variables so that one may apply the differential operator

$$\frac{\partial^m}{(\partial c_1)^{k(J)_1} (\partial c_2)^{k(J)_2} ... (\partial c_n)^{k(J)_n}} \tag{3.2.2}$$

to $v \otimes v \otimes ... \otimes v$, and find

$$\frac{\partial^m (v \otimes v \otimes ... \otimes v)}{(\partial c_1)^{k(J)_1} (\partial c_2)^{k(J)_2} ... (\partial c_n)^{k(J)_n}} = \iota_m(\mathbf{e}_J).$$

The action of 3.2.2 on $v \otimes v \otimes ... \otimes v$ is, by definition, built from a composition of limits. Thus

$$\iota_m(\mathbf{e}_J) \in \operatorname{span}\{v \otimes v \otimes ... \otimes v \mid v \in V\}$$

since subspaces in $V^{\bigotimes m}$ are closed topologically. Therefore

$$\operatorname{span}\{v \otimes v \otimes ... \otimes v \mid v \in V\} = S^m(V).$$

□

Corollary 3.2.4. *For all positive integers m,*

$$S^m(\operatorname{Hom}_{\mathbb{C}}(V, V)) = \operatorname{span}\{A \otimes A \otimes ... \otimes A \mid A \in \operatorname{GL}(V)\}.$$

Proof. Apply Lemma 3.2.3 in the setting of $\operatorname{Hom}_{\mathbb{C}}(V, V)$ to find

$$S^m(\operatorname{Hom}_{\mathbb{C}}(V, V)) = \operatorname{span}\{A \otimes A \otimes ... \otimes A \mid A \in \operatorname{Hom}_{\mathbb{C}}(V, V)\}.$$

But by Lemma 3.2.1, $\operatorname{Hom}_{\mathbb{C}}(V, V) = \overline{\operatorname{GL}(V)}$. Furthermore, the subspace

$$\operatorname{span}\{A \otimes A \otimes ... \otimes A \mid A \in \operatorname{GL}(V)$$

is closed, and thus contains the limits of all the sequences of its elements. Therefore, for all $B \in \operatorname{Hom}_{\mathbb{C}}(V, V)$,

$$B \otimes B \otimes ... \otimes B \in \operatorname{span}\{A \otimes A \otimes ... \otimes A \mid A \in \operatorname{GL}(V)\}.$$

□

Theorem 3.2.5. *For each positive integer $m \geq 1$,*

$$\operatorname{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \operatorname{span}\{A^{\otimes m} \mid A \in \operatorname{GL}(V)\}.$$

Proof. Let φ be the linear isomorphism from Lemma 3.2.2. Then for all $A \in \operatorname{GL}(V)$,

$$\varphi(A \otimes A \otimes ... \otimes A) = A^{\otimes m}.$$

Hence,

$$\operatorname{span}\{A^{\otimes m} \mid A \in \operatorname{GL}(V)\} = \varphi(\operatorname{span}\{A \otimes A \otimes ... \otimes A \mid A \in \operatorname{GL}(V)\}).$$

Note that φ is an $\mathcal{S}_m-$ isomorphism between $\operatorname{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m})$ and $\operatorname{Hom}_{\mathbb{C}}(V, V)^{\bigotimes m}$, where $\operatorname{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m})$ is considered an $\mathcal{S}_m-$module under the action

$$(\sigma, T) \to \sigma \circ T \circ \sigma^{-1}.$$

Indeed, let $v_1 \otimes ... \otimes v_m \in V^{\bigotimes m}$, $A_1 \otimes ... \otimes A_m \in \mathrm{Hom}_{\mathbb{C}}(V,V)^{\bigotimes m}$, and $\sigma \in \mathcal{S}_m$. Then

$$\begin{aligned}
(\sigma(\varphi(A_1 \otimes ... \otimes A_m)))\,(v_1 \otimes ... \otimes v_m) &= (\sigma \circ \varphi(A_1 \otimes ... \otimes A_m) \circ \sigma^{-1})(v_1 \otimes ... \otimes v_m) \\
&= (\sigma \circ \varphi(A_1 \otimes ... \otimes A_m))(v_{\sigma(1)} \otimes ... \otimes v_{\sigma(m)}) \\
&= \sigma\left((A_1(v_{\sigma(1)}) \otimes ... \otimes A_m(v_{\sigma(m)})\right) \\
&= A_{\sigma^{-1}(1)}(v_1) \otimes ... \otimes A_{\sigma^{-1}(m)}(v_m).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi(\sigma(A_1 \otimes ... \otimes A_m))(v_1 \otimes ... \otimes v_m) &= \varphi(A_{\sigma^{-1}(1)} \otimes ... \otimes A_{\sigma^{-1}(m)})(v_1 \otimes ... \otimes v_m) \\
&= A_{\sigma^{-1}(1)}(v_1) \otimes ... \otimes A_{\sigma^{-1}(m)}(v_m).
\end{aligned}$$

Therefore φ is a $\mathcal{S}_m-$isomorphism. as claimed.

With this in mind, $\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m})$ and $S^m(\mathrm{Hom}_{\mathbb{C}}(V,V))$ are the isotypic components of the trivial $\mathcal{S}_m-$module in

$$\mathrm{Hom}_{\mathbb{C}}(V^{\bigotimes m}, V^{\bigotimes m}) \quad \text{and} \quad \mathrm{Hom}_{\mathbb{C}}(V,V)^{\bigotimes m},$$

respectively. Hence

$$\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \varphi(S^m(\mathrm{Hom}_{\mathbb{C}}(V,V))).$$

Therefore by Lemma 3.2.3,

$$\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \varphi(S^m(\mathrm{Hom}_{\mathbb{C}}(V,V))) = \mathrm{span}\{A^{\otimes m} \mid A \in \mathrm{GL}(V)\}.$$

□

3.3 $\mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m})$: An irreducible $\mathrm{GL}(V)$-module

By Maschke's Theorem, $V^{\bigotimes m}$ is completely reducible as an $\mathcal{S}_m-$module. In addition, all the distinct irreducible $\mathcal{S}_m-$modules are given by the Specht modules $\mathcal{S}^\lambda$, one for each $\lambda \vdash m$. Using this, one can write the decomposition of $V^{\bigotimes m}$ as

$$V^{\bigotimes m} = \bigoplus_{\lambda \vdash m} W^\lambda$$

where, for each $\lambda \vdash m$, W^λ is the isotypic component corresponding to the Specht module $\mathcal{S}^\lambda$ of multiplicity $m_\lambda \geq 0$.

Moreover, each W^λ is a $\mathrm{GL}(V)-$module as well. Indeed, for any $A \in \mathrm{GL}(V)$ and $\lambda \vdash m$,

$$A^{\otimes m}(W^\lambda) = W^\lambda$$

since $A^{\otimes m} \in \mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m})$. Now the objective is to obtain an irreducible $\mathrm{GL}(V)$-

module from W^λ using the identity

$$\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \mathrm{span}\{A^{\otimes m} \mid A \in \mathrm{GL}(V)\}.$$

This is necessary since one cannot appeal to Maschke's Theorem to decompose W^λ into irreducible $\mathrm{GL}(V)$-submodules, as $\mathrm{GL}(V)$ is not a finite group.

So, suppose that $W^\lambda \neq \{0\}$, i.e. $m_\lambda > 0$. Then by Proposition 1.4.1 and Corollary 1.4.4 respectively,

$$W^\lambda \cong \mathbb{C}^{m_\lambda} \otimes \mathcal{S}^\lambda$$

as $\mathcal{S}_m$-modules, and

$$\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i}, \mathbb{C}^{m_i})$$

as rings. With this in mind, it will be beneficial to illustrate the second isomorphism by analyzing how $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$ interacts with $\mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m})$. Consider the decomposition of W^λ into irreducible $\mathcal{S}_m-$modules

$$W^\lambda = W_1^\lambda \oplus W_2^\lambda \oplus ... \oplus W_{m_\lambda}^\lambda,$$

and let

$$\{\phi_i \mid i \in [m_\lambda]\} \tag{3.3.1}$$

be a collection of $\mathcal{S}_m-$ isomorphisms such that $\phi_i : \mathcal{S}^\lambda \to W_i^\lambda$. In addition, for each $i, j \in [m_\lambda]$, let $\Psi_{ij} \in \mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$ be an $\mathcal{S}_m-$ isomorphism such that

$$\Psi_{ij}(W_j^\lambda) = W_i^\lambda,$$

and $\Psi_{ij}(W_k^\lambda) = \{0\}$ for all $k \neq j$. This collection does exist: One can start with $\Psi_{ij} \in \mathrm{Hom}_{\mathcal{S}_m}(W_j^\lambda, W_i^\lambda)$. Then afterward, lift Ψ_{ij} to all of $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$ by declaring that $\Psi_{ij}(W_k^\lambda) = \{0\}$ for all $k \neq j$.

Now by construction, $\{\Psi_{ij} \mid (i,j) \in [m_\lambda]^2\}$ forms a basis for $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$ since this collection is linearly independent, and $\dim(\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)) = m_\lambda^2$. Furthermore, using a similar argument, $\{\phi_i \mid i \in [m_\lambda]\}$ is a basis for $\mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m})$, where each ϕ_i spans the one dimensional $\mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, W_i^\lambda)$.

Note that $\Psi_{ij} \circ \phi_j \in \mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, W_i^\lambda)$ since the composition

$$\Psi_{ij} \circ \phi_j : \mathcal{S}^\lambda \to W_i^\lambda$$

defines another isomorphism. However, $\Psi_{ij} \circ \phi_j = c_{ij}\phi_i$, for some constant $c_{ij} \in \mathbb{C}$. So, set $\Theta_{ij} = \frac{1}{c_{ij}}\Psi_{ij}$ for each i, j. Then

$$\Theta_{ij} \circ \phi_l = \delta_{jl}\phi_i.$$

The collection

$$\{\Theta_{ij} \mid (i,j) \in [m_\lambda]^2\} \tag{3.3.2}$$

is another basis for $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$. Thus if $\Xi = \sum_{i,j} a_{ij}\Theta_{ij}$ and $\varphi = \sum_i b_i\phi_i$, then

$$\begin{aligned}
\Xi \circ \varphi &= \left(\sum_{i,j=1}^{m_\lambda} a_{ij}\Theta_{ij}\right) \circ \left(\sum_{i=1}^{m_\lambda} b_i\phi_i\right) \\
&= \sum_{i,j,l=1}^{m_\lambda} a_{ij}b_l\left(\Theta_{ij} \circ \phi_l\right) \\
&= \sum_{i,j,l=1}^{m_\lambda} a_{ij}b_l\delta_{jl}\phi_i \\
&= \sum_{i=1}^{m_\lambda}\left(\sum_{l=1}^{m_\lambda} a_{il}b_l\right)\phi_i.
\end{aligned}$$

Therefore, if e_i denotes the ith standard basis element, and $\hat{E}_{ij}$ denotes the (i,j)th transformation unit, i.e.

$$\hat{E}_{ij}(e_l) = \delta_{jl}e_i,$$

then $\mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m}) \cong \mathbb{C}^{m_i}$ and $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda) \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{m_i})$ using the identification,

$$\phi_i \leftrightarrow e_i \quad \text{and} \quad \Theta_{ij} \leftrightarrow \hat{E}_{ij}.$$

Motivated by this is the following definition.

Definition 3.3.1. Let $\lambda \vdash n$, and let W^λ be the isotypic component associated to $\mathcal{S}^\lambda$. The set

$$U^\lambda := \mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m})$$

is the *multiplicity space associated to* W^λ.

It will be shown that U^λ is an irreducible $\mathrm{GL}(V)$-module. But first, it needs to be established that U^λ is indeed a $\mathrm{GL}(V)-$module. Let $A \in \mathrm{GL}(V)$ and $\varphi \in U^\lambda$ Then

$$A^{\otimes m} \circ \varphi \in U^\lambda$$

since the domain of

$$A^{\otimes m} \circ \varphi$$

is $\mathcal{S}^\lambda$, and since the composition of two functions that commute with $\mathcal{S}_m$ also commutes with $\mathcal{S}_m$.

In addition to this, $A^{\otimes m}$ is complex linear. Thus, for all $a, b \in \mathbb{C}$ and $\varphi, \psi \in U^\lambda$,

$$A^{\otimes m} \circ (a\varphi + b\psi) = a(A^{\otimes m} \circ \varphi) + b(A^{\otimes m} \circ \psi).$$

Therefore, post-composition by $A^{\otimes m}$ is also complex linear.

Furthermore, post-composition defines a group action. Indeed. first,

$$T^m I \circ \varphi = (\mathrm{id}_{V^{\bigotimes m}}) \circ \varphi = \varphi.$$

And second, let $A, B \in \mathrm{GL}(V)$, let $\varphi \in U^\lambda \setminus \{0\}$, and suppose that $f \in \mathcal{S}^\lambda$. Then

$$\begin{aligned} ((A^{\otimes m}B) \circ \varphi)(f) &= (A^{\otimes m}B)(\varphi(f)) \\ &= A^{\otimes m}(B^{\otimes m}(\varphi(f))), \end{aligned}$$

where

$$\begin{aligned} (A^{\otimes m} \circ (B^{\otimes m} \circ \varphi))(f) &= A^{\otimes m}((B^{\otimes m} \circ \varphi)(f)) \\ &= A^{\otimes m}(B^{\otimes m}(\varphi(f))). \end{aligned}$$

By Schur's lemma, φ is injective since it is not equal to 0, and $\mathcal{S}^\lambda$ is irreducible. Thus $(A^{\otimes m}B) \circ \varphi$ and $A^{\otimes m} \circ (B^{\otimes m} \circ \varphi)$ are also injective. Hence

$$(A^{\otimes m} \circ (B^{\otimes m} \circ \varphi))(f) = ((A^{\otimes m}B) \circ \varphi)(f),$$

and therefore

$$(A^{\otimes m}B) \circ \varphi) = A^{\otimes m} \circ (B^{\otimes m} \circ \varphi).$$

Now that U^λ is a $\mathrm{GL}(V)-$module, the next lemma will be useful to show that U^λ is also irreducible.

Lemma 3.3.2. *Let V be a $G-$module, and $\rho : G \to \mathrm{GL}(V)$ denote the representation that V carries as a $G-$module. If*

$$\{\rho(g) \in \mathrm{GL}(V) \mid g \in G\}$$

spans all of $\mathrm{Hom}_{\mathbb{C}}(V, V)$, then V is irreducible.

Proof. Suppose that $W \leq V$ is a nonzero $G-$submodule, and let $w \in W$. First, lift w to a basis $\{w, b_1, ..., b_{n-1}\}$ of V. Now, for each $j \in [n-1]$, define $\hat{E}_j \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ by

$$\hat{E}_j(w) := b_j,$$

with $\hat{E}_j(b_i) := 0$ for each $i \in [n-1]$.

By the hypothesis that $\{\rho(g) \in \mathrm{GL}(V) \mid g \in G\}$ spans $\mathrm{Hom}_{\mathbb{C}}(V, V)$, there exists $N_j \geq 1$ and collections $\{g_{ij}\} \subseteq G$ and $\{a_{ij}\} \subseteq \mathbb{C}$ such that

$$\hat{E}_j = \sum_{i=1}^{N_j} a_{ji}\rho(g_{ji})$$

for each j. However, this implies that

$$b_j = \hat{E}_j(w) = \left(\sum_{i=1}^{N_j} a_{ji}\rho(g_{ji})\right) w = \sum_{i=1}^{N_j} (a_{ji}\rho(g_{ji})w).$$

Thus $b_j \in W$ since $w \in W$, and W is a $G-$submodule. Therefore $W = V$. $\square$

Theorem 3.3.3. *Let $\lambda \vdash m$, and suppose W^λ, the isotypic component associated to $\mathcal{S}^\lambda$, is* *nonzero in $V^{\bigotimes m}$. Then the corresponding multiplicity space U^λ is an irreducible* GL(V)-*module.*

Proof. Since $m_\lambda > 0$, let $\{\phi_i \mid i \in [m_\lambda]\}$ be the collection of $\mathcal{S}_m-$ isomorphisms introduced in 3.3.1, and recall from 3.3.2, the basis for $\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda)$, $\{\Theta_{ij} \mid (i,j) \in [m_\lambda]^2\}$, such that

$$\Theta_{ij} \circ \phi_l = \delta_{jl}\phi_i.$$

Now, $\{\phi_i \mid i \in [m_\lambda]\}$ is a basis for U^λ since by definition, $U^\lambda = \mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^\lambda, V^{\bigotimes m})$. With this in mind, note that

$$\mathrm{Hom}_{\mathcal{S}_m}(W^\lambda, W^\lambda) \cong \mathrm{Hom}_{\mathbb{C}}(U^\lambda, U^\lambda),$$

which is established by use of the action of $\{\Theta_{ij} \mid (i,j) \in [m_\lambda]^2\}$ on $\{\phi_i \mid i \in [m_\lambda]\}$ stated above. Furthermore, by Theorem 3.2.5,

$$\mathrm{Hom}_{\mathcal{S}_m}(V^{\bigotimes m}, V^{\bigotimes m}) = \mathrm{span}\{A^{\otimes m} \mid A \in \mathrm{GL}(V)\}.$$

Thus for each Θ_{ij}, there exists an $N_{ij} \geq 1$ with collections $\{A_{lij}\} \subseteq \mathrm{GL}(V)$ and $\{a_{lij}\} \subseteq \mathbb{C}$ such that

$$\Theta_{ij} = \sum_{l=1}^{N_{ij}} a_{lij} A_{lij}^{\otimes m}.$$

In other words, if $\rho : \mathrm{GL}(V) \to \mathrm{GL}(U^\lambda)$ is the equivalent representation that U^λ carries under the action

$$(A, \varphi) \to A^{\otimes m} \circ \varphi,$$

then $\{\rho(A) \in \mathrm{GL}(U^\lambda) \mid A \in \mathrm{GL}(V)\}$ spans all of $\mathrm{Hom}_{\mathbb{C}}(U^\lambda, U^\lambda)$. Therefore U^λ is an irreducible GL$(V)-$module by use of Lemma 3.3.2. □

3.4 Realizations using the general projection operators

For each $\lambda \vdash m$, one will be able to construct an irreducible tensor representation of GL(V) using the multiplicity space U^λ whenever the isotypic component W^λ associated to $\mathcal{S}^\lambda$ is nonzero. Considering Section 1.4, if $W^\lambda = 0$, then naturally $U^\lambda = 0$. In this situation, no corresponding irreducible tensor representation of GL(V) appears. There is a necessary and sufficient condition to determine whether or not $U^\lambda = 0$; however, this will be addressed in Chapter 6. (See Corollary 6.3.3.)

Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$. The *standard tableau of type* λ, denoted t_λ, is the tableau such that

$$(t_\lambda)_{1,j} = j,$$

for all $j \in [\lambda_1]$, with

$$(t_\lambda)_{i,j} = (\lambda_1 + \lambda_2 + ... + \lambda_{i-1}) + j,$$

for all $2 \leq i \leq l$, and all $j \in [\lambda_i]$. Intuitively, t_λ is obtained by listing the integers of $[m]$ in their natural order successively in the rows of the Ferrers diagram of the partition λ. Considering Section 2.6, the general projection operator of type t_λ naturally defines an element of $\mathrm{Hom}_{\mathrm{GL}(V)}(V^{\bigotimes m}, V^{\bigotimes m})$ via the assignment

$$\mathbf{v} \to \epsilon_{t_\lambda}(\mathbf{v}),$$

for all $\mathbf{v} \in V^{\bigotimes m}$. (More generally, any $F \in \mathbb{C}[\mathcal{S}_m]$ defines one such element.) Consequently,

$$\epsilon_{t_\lambda}(V^{\bigotimes m}),$$

the image of $V^{\bigotimes m}$ under ϵ_{t_λ}, is a $\mathrm{GL}(V)$-submodule in $V^{\bigotimes m}$.

For the following lemma, note the following isomorphism of $\mathrm{GL}(V)$-modules

$$U^\lambda \cong \mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m}),$$

where the action of $\mathrm{GL}(V)$ on $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$ is given by post-composition. (For proof, consider $\mathcal{S}^\lambda \cong \langle \epsilon_{t_\lambda} \rangle_L$, as $\mathcal{S}_m$-modules.)

Lemma 3.4.1. *Let $\mathbf{v} \in V^{\bigotimes m}$. Then the assignment,*

$$\mathrm{eval}_{\mathbf{v}}(F) := F(\mathbf{v})$$

for $F \in \langle \epsilon_{t_\lambda} \rangle_L$, defines an $\mathcal{S}_m$-homomorphism from $\langle \epsilon_{t_\lambda} \rangle_L$ to $V^{\bigotimes m}$.

Moreover, the assignment

$$\mathbf{v} \mapsto \mathrm{eval}_{\mathbf{v}}$$

is a $\mathrm{GL}(V)$-homomorphism from $V^{\bigotimes m}$ to $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$ for each $\mathbf{v} \in V^{\bigotimes m}$.

Proof. It is straightforward to see that evaluations of linear maps are themselves linear. So let $\mathbf{v} \in V^{\bigotimes m}$, and $F \in \langle \epsilon_{t_\lambda} \rangle_L$. Then, for $\sigma \in \mathcal{S}_m$,

$$\mathrm{eval}_{\mathbf{v}}(\sigma F) = (\sigma F)(\mathbf{v}) = \sigma(F(\mathbf{v})) = \sigma(\mathrm{eval}_{\mathbf{v}}(F)).$$

Thus $\mathrm{eval}_{\mathbf{v}} \in \mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$.

Now, define

$$E : V^{\bigotimes m} \to \mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$$

by

$$E(\mathbf{v}) := \mathrm{eval}_{\mathbf{v}}.$$

Again, it is easy to verify that this map is linear. So, still considering the elements declared above, let $A \in \mathrm{GL}(V)$. Then

$$\mathrm{eval}_{A^{\otimes m}(\mathbf{v})}(F) = F(A^{\otimes m}(\mathbf{v})) = A^{\otimes m}(F(\mathbf{v})) = A^{\otimes m}(\mathrm{eval}_{\mathbf{v}}(F)).$$

Therefore,

$$E(A^{\otimes m}(\mathbf{v})) = \mathrm{eval}_{A^{\otimes m}(\mathbf{v})} = A^{\otimes m} \circ \mathrm{eval}_{\mathbf{v}} = A^{\otimes m} \circ E(\mathbf{v}).$$

□

Theorem 3.4.2. *Let $\lambda \vdash m$. Suppose that W^λ, the isotypic component associated to $\mathcal{S}^\lambda$, is* *nonzero in* $V^{\bigotimes m}$*. Then, as* $\mathrm{GL}(V)$*-modules,*

$$U^\lambda \cong \epsilon_{t_\lambda}(V^{\bigotimes m}).$$

Proof. First, the $\mathrm{GL}(V)$-homomorphism defined in Lemma 3.4.1,

$$\mathbf{v} \mapsto \mathrm{eval}_{\mathbf{v}},$$

is surjective. Indeed, $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$ is irreducible since $W^\lambda \neq 0$. Thus by Corollary 1.3.4, the result follows.

Consequently, for some collection $\{\mathbf{v}_i \in V^{\bigotimes m} \mid i \in [\dim U^\lambda]\}$,

$$\{\mathrm{eval}_{\mathbf{v}_i} \mid i \in [\dim U^\lambda]\}$$

is a basis for $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$. Let $\phi \in \mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$. Considering Lemma 3.4.1,

$$\mathrm{eval}_{\epsilon_{t_\lambda}}(\phi) := \phi(\epsilon_{t_\lambda})$$

is a $\mathrm{GL}(V)$-homomorphism from $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})$ to $V^{\bigotimes m}$. In fact, this assignment is nonzero since $\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m}) \neq 0$. Thus by Schur's Lemma, it's a $\mathrm{GL}(V)$-isomorphism as well. $\square$

With these last two points, note that

$$\epsilon_{t_\lambda}(\mathbf{v}) = \mathrm{eval}_{\mathbf{v}}(\epsilon_{t_\lambda}) = \mathrm{eval}_{\epsilon_{t_\lambda}}(\mathrm{eval}_{\mathbf{v}})$$

for any $\mathbf{v} \in V^{\bigotimes m}$. Thus

$$\mathrm{eval}_{\epsilon_{t_\lambda}}(\mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m})) = \epsilon_{t_\lambda}(V^{\bigotimes m}).$$

Therefore, since $\mathrm{eval}_{\epsilon_{t_\lambda}}$ is a $\mathrm{GL}(V)$-isomorphism,

$$U^\lambda \cong \mathrm{Hom}_{\mathcal{S}_m}(\langle \epsilon_{t_\lambda} \rangle_L, V^{\bigotimes m}) \cong \epsilon_{t_\lambda}(V^{\bigotimes m})$$

as $\mathrm{GL}(V)$-modules.

3.5 $(\mathbb{C}^n)^{\bigotimes m}$ as a $\mathrm{GL}(n, \mathbb{C})$-module

With the standard basis, and the use of matrix multiplication on column vectors in $\mathbb{C}^n$, one sees $(\mathbb{C}^n)^{\bigotimes m}$ as a $\mathrm{GL}(n, \mathbb{C})$-module in a completely equivalent fashion as a $\mathrm{GL}(\mathbb{C}^n)$-module. In fact, all the theory established in this chapter will be applied to the setting of $(\mathbb{C}^n)^{\bigotimes m}$ as a group module for $\mathrm{GL}(n, \mathbb{C})$, since the irreducible tensor representations of $\mathrm{GL}(n, \mathbb{C})$ carried there provide all the irreducible representations for the subgroup $\mathrm{SU}(n)$.

Chapter 4

Finite-dimensional representations of $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{SU}(n)$

It has been claimed that all finite dimensional irreducible representations of $\mathrm{SU}(n)$ can be realized on the various tensor powers of $\mathbb{C}^n$. Two questions must be addressed if this is to be true: how can it be guaranteed that restricting the irreducible tensor representations of $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{SU}(n)$ also yields irreducible representations for the subgroup? And, how can it be assured that this method captures all the finite dimensional irreducible representations for $\mathrm{SU}(n)$? The answers are mediated by a third matrix group, the special linear group $\mathrm{SL}(n, \mathbb{C})$. In addition, it will be necessary to employ the analytical structure that these matrix groups possess as *Lie groups*. This must occur, as the irreducible tensor representations are examples of Lie group representations, carrying more structure such as continuity and various degrees of differentiability. This chapter will establish relevant representation theory for matrix Lie groups, and that the finite-dimensional irreducible continuous representations of $\mathrm{SU}(n)$ are in one to one correspondence with complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$. The importance of this result will reveal itself in Chapters 5 and 6, where it will be established that the complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$ are themselves in complete correspondence with integer partitions characterizing the irreducible tensor representations of $\mathrm{GL}(n, \mathbb{C})$. The following presentation on matrix Lie group is modeled after the treatment provided by B. Hall [1].

The vector spaces are assumed complex and finite-dimensional, "finite representation" of a matrix group will mean "finite-dimensional," and furthermore, topological concepts concerning $M_n(\mathbb{C})$ will be relative to the euclidean topology on $M_n(\mathbb{C})$ induced by norm $||\cdot||$, where

$$||X|| := \left(\sum_{k,l=1}^{n} |x_{kl}|^2 \right)^{\frac{1}{2}}$$

for all $X \in M_n(\mathbb{C})$.

4.1 Elementary matrix Lie group theory

4.1.1 The special linear group and the special unitary group

Let n be a positive integer. The *(complex) special linear group of degree* n, $\mathrm{SL}(n, \mathbb{C})$, is the set of complex $n \times n$ matrices with determinant one. It is known that matrices with nonzero determinant are invertible. Therefore $\mathrm{SL}(n, \mathbb{C})$ forms a subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Consider $\langle \cdot \mid \cdot \rangle$, the standard inner product on $\mathbb{C}^n$. The group of *unitary matrices of degree* n, denoted $\mathrm{U}(n)$, is the set of matrices that preserve the inner product, $\langle \cdot \mid \cdot \rangle$, i.e.

$$\langle Av, \mid Aw \rangle = \langle v \mid w \rangle,$$

for each $A \in \mathrm{U}(n)$ and all $v, w \in \mathbb{C}^n$. The group of *special unitary matrices of degree* n, $\mathrm{SU}(n)$, consists of all unitary matrices with determinant equal to 1. In other words,

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$$

and in particular, $\mathrm{SU}(n) \subseteq \mathrm{SL}(n, \mathbb{C})$. An alternative description of $\mathrm{SU}(n)$ is

$$\mathrm{SU}(n) = \{A \in M_n(\mathbb{C}) \mid AA^\dagger = I, \det A = 1\},$$

where $A^\dagger$ denotes the conjugate transpose of $A \in M_n(\mathbb{C})$.

4.1.2 $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{SU}(n)$ as matrix Lie groups

Pleasantly enough, matrix Lie groups have simple origins. It might not apparent, but they are, in fact, smooth manifolds (some being complex manifolds). The following definition avoids the language of manifolds; however, this connection will be established later in Section 4.4.

For the following, consider the subspace topology on $\mathrm{GL}(n, \mathbb{C})$, inherited from the euclidean topology on $M_n(\mathbb{C})$.

Definition 4.1.1. A *matrix Lie group* is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$.

By the definition, $\mathrm{GL}(n, \mathbb{C})$ is clearly a matrix Lie group. In addition, $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ are also matrix Lie groups. For starters, being a polynomial map in the matrix entries,

$$\det : M_n(\mathbb{C}) \to \mathbb{C}$$

defines a continuous function. By the definition of $\mathrm{SL}(n, \mathbb{C})$,

$$\mathrm{SL}(n, \mathbb{C}) = \det^{-1}\{1\}.$$

Thus, as the inverse image of a closed set under a continuous function, $\mathrm{SL}(n, \mathbb{C})$ is a closed subset of $M_n(\mathbb{C})$. However $\mathrm{SL}(n, \mathbb{C})$ is already contained in $\mathrm{GL}(n, \mathbb{C})$. Therefore $\mathrm{SL}(n, \mathbb{C})$ a matrix Lie group.

On the other hand, The assignment

$$A \mapsto AA^{\dagger}$$

is continuous. Thus $\mathrm{U}(n) = \{A \in M_n(\mathbb{C}) \mid AA^{\dagger} = I_n\}$ is closed subset of $M_n(\mathbb{C})$. Consequently, being the intersection of two closed subsets, $\mathrm{SU}(n)$ is closed as well. Therefore $\mathrm{SU}(n)$ is a matrix Lie group since it a subgroup of $\mathrm{GL}(n, \mathbb{C})$ that is closed in $M_n(\mathbb{C})$.

Matrix Lie groups form a subset of Lie groups in general. For reference, here is the definition of an arbitrary Lie group.

Definition 4.1.2. A *(complex) Lie group* G is a group that is also a (complex) smooth manifold such that

$$\begin{array}{rcl} G \times G & \to & G \\ (x, y) & \mapsto & x^{-1}y \end{array}$$

is a (complex analytic) smooth map between (complex) smooth manifolds.

4.1.3 The matrix exponential and logarithm

Interestingly enough, one can extend the definition of the real/complex exponential to the setting of $M_n(\mathbb{C})$. This construct will be vital in the developments of theory matrix Lie groups. For example, due to the fact that the Lie groups of consideration are all matrix Lie groups, one will define the Lie algebra in terms of one parameter subgroups using the exponential map.

Definition 4.1.3. The *exponential map*, or *matrix exponential* of $X \in M_n(\mathbb{C})$ is defined by

$$\mathrm{e}^X := I + X + \frac{X^2}{2} + ... + \frac{X^k}{k!} + ... \, .$$

Remark. It was assumed that $M_n(\mathbb{C})$ is a valid domain for the matrix exponential. This is indeed true. Since

$$\left|\left| \sum_{k=1}^{K} \frac{X^k}{k!} \right|\right| \leq \sum_{k=1}^{K} \frac{||X||^k}{k!} \leq \mathrm{e}^{||X||}$$

for each $K \geq 1$, e^X is convergent for all $X \in M_n(\mathbb{C})$. Furthermore, note that the matrix exponential is continuous on $M_n(\mathbb{C})$ (See the proof of Proposition 4.1.5).

Having a significantly more restricted domain, is the related matrix logarithm.

Definition 4.1.4. Let $A \in M_n(\mathbb{C})$. Then whenever convergence occurs, the *matrix logarithm of A* is defined by

$$\log(A) := \sum_{k=1}^{\infty} (-1)^{m+1} \frac{(A - I)^k}{k}.$$

The following proposition shows that the matrix logarithm provides a continuous local inverse to the exponential map.

Proposition 4.1.5. *Let $A \in M_n(\mathbb{C})$. Then the function*

$$A \mapsto \log(A)$$

exists, is continuous, and

$$\mathrm{e}^{\log(A)} = A$$

whenever $||A - I|| < 1$. Moreover, if $||A|| < \ln 2$, then $||\mathrm{e}^A - I|| < 1$ and

$$\log(\mathrm{e}^A) = A.$$

Proof. See the proof found in [1]. However, the basic idea is to establish the proposition for diagonalizable matrices. Then use the fact that these matrices are dense in $M_n(\mathbb{C})$ since the matrix exponential and logarithm are continuous operations. To see continuity, note that uniform convergence of continuous partial sums is occurring for each operation on the sets $\{X \in M_n(\mathbb{C}) \mid ||X|| \leq R\}$ of appropriate $R > 0$. This shows continuity on these 0 for each series □

In addition to continuity, the exponential map and matrix logarithm are smooth (where appropriate). Consider the following definition concerning differentiability of curves.

Definition 4.1.6. Let I be an open interval, $\gamma : I \to M_n(\mathbb{C})$ be a matrix valued function, and suppose $t \in I$. If the limit exists, then the derivative of γ at t is defined by

$$\frac{d\gamma(t)}{dt} := \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

Note that γ is differentiable at t if and only if, for all $1 \leq i, j \leq n$, γ_{ij} is differentiable at t. Furthermore, if γ is differentiable on all of its domain, then γ is *smooth* whenever each γ_{ij} is smooth.

Proposition 4.1.7. *For each $X \in M_n(\mathbb{C})$,*

$$t \to \mathrm{e}^{tX}$$

defines a smooth curve on $\mathbb{R}$, with

$$\frac{d}{dt}\,\mathrm{e}^{tX} = X\,\mathrm{e}^{tX}.$$

Proof. The entries of e^{tX} are given by convergent power series in the variable t. Indeed, for each $i, j \in [n]$,

$$(\mathrm{e}^{tX})_{i,j} = \sum_{k=0}^{\infty} \frac{t^k (X^k)_{i,j}}{k!}.$$

Thus e^{tX} defines a smooth function from $\mathbb{R}$ into $M_n(\mathbb{C})$. Now one is able to differentiate convergent power series term-by-term. Therefore the result follows. For more detail see [1]. □

Definition 4.1.8. A continuous map $\gamma : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ is a *one parameter subgroup* of $\mathrm{GL}(n, \mathbb{C})$ if

(1) $\gamma(0) = I$, and

(2) $\gamma(t + s) = \gamma(t)\gamma(s)$, for all $t, s \in \mathbb{R}$.

In other words, one parameter subgroups are continuous group homomorphisms from $\mathbb{R}$ into $\mathrm{GL}(n, \mathbb{C})$. Importantly, these objects are well behaved in the sense that they must take the form of a smooth curve outlined in Proposition 4.1.7.

Proposition 4.1.9. *Let* $\gamma : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ *be a one parameter subgroup of* $\mathrm{GL}(n, \mathbb{C})$. *Then there exists a unique* $X \in M_n(\mathbb{C})$ *such that*

$$\gamma(t) = \mathrm{e}^{tX}$$

for all t.

Proof. Uniqueness is immediate by considering Proposition 4.1.7. Indeed, if there is an $X \in M_n(\mathbb{C})$ such that $\gamma(t) = \mathrm{e}^{tX}$ for all t, then

$$X = \frac{d}{dt} \left(\gamma(t)\right)\Big|_{t=0}$$

To see existence, let $\varepsilon < \ln 2$, define

$$N_{\frac{\varepsilon}{2}} := \left\{ X \in M_n(\mathbb{C}) \,\middle|\, ||X|| < \frac{\varepsilon}{2} \right\},$$

and set $N_I := \{\mathrm{e}^X \mid X \in N_{\frac{\varepsilon}{2}}\}$. If γ is a one parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$, then by continuity of γ, there exists a $\delta > 0$ such that

$$\gamma(t) \in N_I$$

for all $|t| \leq \delta$. As a result, $\log \gamma(t)$ will be defined for such values of t.

Consider the element

$$X = \frac{1}{\delta} \log \gamma(\delta).$$

First, $\gamma(\delta) = \mathrm{e}^{\delta X}$ since $\delta X = \log \gamma(\delta)$. Furthermore, $\gamma(\frac{\delta}{2}) \in N_I$, and thus

$$\log \gamma \left(\frac{\delta}{2}\right) \in N_{\frac{\varepsilon}{2}}.$$

Consequently $2 \log \gamma(\frac{\delta}{2}) \in N_\varepsilon$ since

$$\left|\left|2 \log \gamma \left(\frac{\delta}{2}\right)\right|\right| < 2\frac{\varepsilon}{2} = \varepsilon.$$

Now, as a one parameter subgroup, $\gamma(t)^2 = \gamma(2t)$ for any t. Thus

$$\mathrm{e}^{2\log\gamma(\frac{\delta}{2})} = \gamma\left(\frac{\delta}{2}\right)^2 = \gamma\left(2\frac{\delta}{2}\right) = \gamma(\delta) = \mathrm{e}^{\delta X}.$$

However, since $\mathrm{e}|_{N_\varepsilon}$ is injective, $2\log\gamma(\frac{\delta}{2}) = \delta X$. Hence $\log\gamma(\frac{\delta}{2}) = \frac{\delta X}{2}$, and $\gamma(\frac{\delta}{2}) = \mathrm{e}^{\frac{\delta}{2}X}$. By repeating this argument with $\gamma(\frac{\delta}{4})$, one will find that $\log\gamma(\frac{\delta}{4}) = \frac{\delta X}{4}$, and consequently $\gamma(\frac{\delta}{4}) = \mathrm{e}^{\frac{\delta}{4}X}$. Continuing,

$$\gamma\left(\frac{\delta}{2^k}\right) = \mathrm{e}^{(\frac{\delta}{2^k})X}$$

for all positive integers k. Thus for any integer m,

$$\gamma\left(m\frac{\delta}{2^k}\right) = \gamma\left(\frac{\delta}{2^k}\right)^m = \mathrm{e}^{(m\frac{\delta}{2^k})X}.$$

Finally, $\mathbb{D} = \{m\frac{\delta}{2^k} \mid m \in \mathbb{Z},\, k \in \mathbb{N}\}$ is dense in $\mathbb{R}$, and

$$\gamma(t) = \mathrm{e}^{tX}$$

for all $t \in \mathbb{D}$. Therefore $\gamma(t) = \mathrm{e}^{tX}$ for all $t \in \mathbb{R}$ since they agree on a dense subset. □

4.1.4 The matrix Lie algebra

The Lie algebras of $\mathrm{SU}(n)$ and $\mathrm{SL}(n,\mathbb{C})$ will prove to be an invaluable tool in connecting properties between the Lie group representations of the two matrix Lie groups. In addition to this, by appealing to the Lie algebra, one will be able to establish the smoothness and analyticity of Lie group representations for $\mathrm{SU}(n)$ and $\mathrm{SL}(n,\mathbb{C})$, respectively. In the language of manifolds, the Lie algebra is the tangent space at the identity element of the Lie group. In this treatment however, considering Proposition 4.1.9, the matrix Lie algebra will be defined equivalently in terms of one parameter subgroups.

Definition 4.1.10. Let G be a matrix Lie group. Its *Lie algebra*, denoted $\mathfrak{g}$, is defined by the following set

$$\mathfrak{g} := \{X \in M_n(\mathbb{C}) \mid \mathrm{e}^{tX} \in G,\, \forall t \in \mathbb{R}\}.$$

For a simple illustration, consider $\mathrm{GL}(n,\mathbb{C})$ itself. Its Lie algebra is denoted $\mathfrak{gl}(n,\mathbb{C})$. For any $X \in M_n(\mathbb{C})$, e^X is invertible with

$$\left(\mathrm{e}^X\right)^{-1} = \mathrm{e}^{-X}.$$

Therefore

$$\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C}).$$

Theorem 4.1.11. *Let G be a matrix Lie group, and X, Y be in $\mathfrak{g}$. Then*

(1) *$rX + sY \in \mathfrak{g}$, for all real numbers r and s, and*

(2) $[X,Y] \in \mathfrak{g}$

Proof. This proof will make use of Lie's product formula:

$$\mathrm{e}^{X+Y} = \lim_{m\to\infty} \left(\mathrm{e}^{\frac{X}{m}}\,\mathrm{e}^{\frac{Y}{m}}\right)^m .$$

Let $r, s \in \mathbb{R}$. If $X, Y \in \mathfrak{g}$, then clearly rX and sY are as well. In particular, if m is a positive integer, then $\frac{tX}{m}$ and $\frac{tY}{m}$ are in $\mathfrak{g}$ for any real t. Therefore

$$\left(\mathrm{e}^{\frac{tX}{m}}\,\mathrm{e}^{\frac{tY}{m}}\right)^m \in G$$

since, by definition of $\mathfrak{g}$, both $\mathrm{e}^{\frac{tX}{m}}$ and $\mathrm{e}^{\frac{tY}{m}}$ are in G. Now, matrix Lie groups are closed subgroups of $\mathrm{GL}(n,\mathbb{C})$, and e^{tX+tY} is invertible. Thus

$$\mathrm{e}^{t(X+Y)} = \mathrm{e}^{tX+tY} = \lim_{m\to\infty} \left(\mathrm{e}^{\frac{tX}{m}}\,\mathrm{e}^{\frac{tY}{m}}\right)^m \in G.$$

Therefore

$$X + Y \in \mathfrak{g}.$$

Finally, suppose that $A \in G$, for each real t,

$$\mathrm{e}^{t(AXA^{-1})} = A\,\mathrm{e}^{tX}\,A^{-1}.$$

Thus $AXA^{-1} \in \mathfrak{g}$. Using this with the identity

$$[X,Y] = \frac{d}{dt}\left(\mathrm{e}^{tX}\,Y\,\mathrm{e}^{-tX}\right)\Big|_{t=0},$$

shows that $[X,Y] \in \mathfrak{g}$. Indeed, $\mathfrak{g}$ is a closed subset of $M_n(\mathbb{C})$. □

By Theorem 4.1.11, matrix Lie algebras equipped with the matrix commutator are *real Lie algebras* in the abstract sense, i.e. a real vector space endowed with an anti-symmetric bilinear product satisfying Jacobi's identity. However for $\mathrm{GL}(n,\mathbb{C})$, $\mathfrak{gl}(n,\mathbb{C})$ is clearly a complex vector space. In other words, $\mathfrak{gl}(n,\mathbb{C})$ is a *complex* Lie algebra, a Lie algebra that is also a complex-linear subspace of $\mathfrak{gl}(n,\mathbb{C})$. Moreover, $\mathrm{SL}(n,\mathbb{C})$ too has a complex Lie algebra, seen by the following theorem.

Theorem 4.1.12. *Let $\mathfrak{sl}(n,\mathbb{C})$ denote the Lie algebra to the matrix Lie group* $\mathrm{SL}(n,\mathbb{C})$. *Then*

$$\mathfrak{sl}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}.$$

Furthermore, $\mathfrak{sl}(n,\mathbb{C})$ is a complex Lie algebra.

Proof. This proof will be centered on the following identity. For any $X \in M_n(\mathbb{C})$,

$$\det(\mathrm{e}^X) = \mathrm{e}^{\mathrm{tr}(X)}.$$

Thus if X satisfies $\mathrm{tr}(X) = 0$, then

$$\det(\mathrm{e}^{tX}) = \mathrm{e}^{\mathrm{tr}(tX)} = 1$$

for all real t. Therefore $X \in \mathfrak{sl}(n, \mathbb{C})$.

Conversely, suppose $X \in \mathfrak{sl}(n, \mathbb{C})$. Then

$$\det(\mathrm{e}^{tX}) = 1$$

for all real t. Thus

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\det(\mathrm{e}^{tX})\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\mathrm{e}^{\mathrm{tr}(X)t}\right)\Big|_{t=0} \\ &= \mathrm{tr}(X). \end{aligned}$$

Therefore

$$\mathfrak{sl}(n, \mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}.$$

Finally, $\mathfrak{sl}(n, \mathbb{C})$ is a complex Lie algebra since the trace of a matrix is a complex-linear operation, and the kernel of a $\mathbb{C}$-linear map from $\mathfrak{gl}(n, \mathbb{C})$ to $\mathbb{C}^r$ equivalently defines a complex-linear subspace of $\mathfrak{gl}(n, \mathbb{C})$. $\square$

Like $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, matrix Lie groups having complex Lie algebras are called *complex* matrix Lie groups.

Theorem 4.1.13. *Let $\mathfrak{su}(n)$ denote the Lie algebra of the matrix Lie group* $\mathrm{SU}(n)$. *Then*

$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^\dagger = -X,\ \mathrm{tr}(X) = 0\}.$$

Proof. By definition,

$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid \mathrm{e}^{tX} \in \mathrm{SU}(n),\ \forall t \in \mathbb{R}\}.$$

Since $\mathrm{SU}(n)$ is a subgroup of $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{tr}(X) = 0$ whenever $X \in \mathfrak{su}(n)$. Two more useful identities will be used,

$$\mathrm{e}^{X^\dagger} = (\mathrm{e}^X)^\dagger,$$

and secondly,

$$\mathrm{e}^{X+Y} = \mathrm{e}^X\,\mathrm{e}^Y$$

whenever $XY = YX$. Suppose $X^\dagger = -X$. Then $XX^\dagger = X^\dagger X$. Thus for each real t,

$$\begin{aligned} \mathrm{e}^{tX}(\mathrm{e}^{tX})^\dagger &= \mathrm{e}^{tX}\,\mathrm{e}^{tX^\dagger} \\ &= \mathrm{e}^{t(X+X^\dagger)} \\ &= \mathrm{e}^{t(X-X)} \\ &= I. \end{aligned}$$

Therefore $X \in \mathrm{U}(n)$.

Conversely, suppose $\mathrm{e}^{tX} \in \mathrm{U}(n)$ for each real t, and define $\gamma(t) = \mathrm{e}^{tX}(\mathrm{e}^{tX})^{\dagger}$. Then, for each t, $\gamma(t) = I$. Thus

$$\begin{aligned} 0 &= \frac{d}{dt}\,\gamma(t)\big|_{t=0} \\ &= \left(\frac{d}{dt}\,\mathrm{e}^{tX}\Big|_{t=0}\right)(\mathrm{e}^{0X})^{\dagger} + \mathrm{e}^{0X}\left(\frac{d}{dt}\,(\mathrm{e}^{tX})^{\dagger}\Big|_{t=0}\right) \\ &= X + X^{\dagger}. \end{aligned}$$

Consequently $X^{\dagger} = -X$. Therefore,

$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^{\dagger} = -X,\, \mathrm{tr}(X) = 0\}.$$

□

Remark. If $X^{\dagger} = -X$, then X is called a *skew-hermitian matrix.*

It is relevant to point out that while $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$ are complex Lie algebras, $\mathfrak{su}(n)$ is strictly real. To see this, suppose $X \in \mathfrak{su}(n)$. Then

$$(iX)^{\dagger} = (-i)(-X) = iX.$$

As a consequence, $iX \in \mathfrak{su}(n)$ if and only if $X = 0$. Therefore $\mathfrak{su}(n)$ is not a complex Lie algebra, and $\mathrm{SU}(n)$ is not a complex matrix Lie group.

4.2 Representations of matrix Lie groups and Lie algebras

Considering the analytic structure that Lie groups possess, a natural question arises. What is an appropriate definition of a Lie group representation? Well, in order to see interesting results by use of the Lie algebra, it definitely should incorporate continuity. But is continuity enough? That is, should differentiability also be assumed in the definition? Remarkably, it turns out that that assuming differentiability is unnecessary, as smoothness of Lie group representations results just from the assumption of continuity. However, a representation being complex analytic is a special case. There are simple examples of continuous representations failing to be complex analytic maps. This section defines these objects and introduces representations of Lie algebras with the modules that carry them.

Note that finite-dimensional representation will be assumed in the following definitions concerning Lie groups and Lie algebras since these representations will be the only ones of interest to the work here.

Definition 4.2.1. A *representation of a matrix Lie group* $\rho : G \to \mathrm{GL}(V)$ is a representation of G as an abstract group that is also a continuous map. A representation of a complex Lie group $\rho : G \to \mathrm{GL}(V)$ is *complex analytic* if the entries of $\rho(A)$ depend analytically on the matrix entries of $A \in G \subseteq \mathrm{GL}(n,\mathbb{C})$, In addition, a *matrix representation* for G is a continuous group homomorphism $\rho : G \to \mathrm{GL}(m,\mathbb{C})$. A *complex analytic* matrix representation is defined similarly.

Recall that $\mathrm{GL}(V)$ can be identified with $\mathrm{GL}(n,\mathbb{C})$. So, let $\mathfrak{gl}(V)$ denote the Lie algebra to $\mathrm{GL}(V)$. A *real or complex Lie algebra homomorphism* is a real or complex linear map that preserves the Lie bracket.

Definition 4.2.2. Let $\mathfrak{g}$ be a complex Lie algebra, and V be a complex vector space. A *complex-linear Lie algebra representation* is a complex Lie algebra homomorphism

$$p : \mathfrak{g} \to \mathfrak{gl}(V).$$

Furthermore, the vector space V is a *(left)* $\mathfrak{g}$*-module* whenever there is an binary operation from $\mathfrak{g} \times V$ to V

$$(X, v) \mapsto Xv$$

such that, for all $v, w \in V$; $X, Y \in \mathfrak{g}$; and complex numbers $a, b \in \mathbb{C}$,

(1) $X(av + bw) = a(Xv) + b(Xw)$,

(2) $(aX + bY)v = a(Xv) + b(Yv)$, and

(3) $[X, Y]v = X(Yv) - Y(Xv)$.

If $\mathfrak{g}$ is a real Lie algebra, then one defines a $\mathfrak{g}$-module by restricting the scalars from $\mathbb{C}$ to $\mathbb{R}$ in condition (2) of Definition 4.2.2. Similarly, one obtains a *(general) complex Lie algebra representation* for the real Lie algebra $\mathfrak{g}$ by requiring

$$p : \mathfrak{g} \to \mathfrak{gl}(V)$$

to be a real Lie algebra homomorphism. From this point on "Lie algebra representation" will mean "(general) complex Lie algebra representation."

Like the case with groups, there is a bijective correspondence between (complex-linear) Lie algebra representations of $\mathfrak{g}$ and $\mathfrak{g}-$modules. Indeed, if $p : \mathfrak{g} \to \mathfrak{gl}(V)$ is a (complex-linear) Lie algebra representation, then V becomes a $\mathfrak{g}-$module by the assignment

$$(X, v) \mapsto p(X)v.$$

Conversely, conditions (1) and (2) for the binary operation in Definition 4.2.2 define a (complex-linear) Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ whenever V is a $\mathfrak{g}-$module.

Some proofs can be simplified by just considering matrices, which warrants the following definition.

Definition 4.2.3. Let $\mathfrak{g}$ be a complex or real Lie algebra. A *(complex-linear) Lie algebra matrix representation* is a (complex) real Lie algebra homomorphism

$$p : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C}).$$

Remark. Note that terms like *irreducible*, *invariant*, $\mathfrak{g}$*-isomorphic*, and $\mathfrak{g}$*-submodule* are analogously defined for representations and matrix representations of Lie algebras. Even Schur's Lemma is valid as well.

Proposition 4.2.4. *Let G be a matrix Lie group, and let $\rho : G \to \mathrm{GL}(V)$ be a representation for G. Then there exists a unique Lie algebra representation*

$$\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(V)$$

such that

$$\rho(\mathrm{e}^X) = \mathrm{e}^{\dot{\rho}(X)}$$

for all $X \in \mathfrak{g}$. Moreover, the representation $\dot{\rho}$ is explicitly given by

$$\dot{\rho}(X) = \frac{d}{dt} \left(\rho(\mathrm{e}^{tX})\right)\Big|_{t=0}.$$

Proof. Since V can be identified with $\mathbb{C}^n$, and likewise, $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{C})$, it will suffice to assume that one is dealing with a matrix representation.

Let $X \in \mathfrak{g}$, and define $\gamma_X : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ by

$$\gamma_X(t) = \rho(\mathrm{e}^{tX}).$$

Clearly, γ_X is continuous and $\gamma_X(0) = I$. Furthermore, $\gamma_X(t+s) = \gamma_X(t)\gamma_X(s)$ for all $s, t \in \mathbb{R}$. To see this, ρ is a group homomorphism, and $\mathrm{e}^{(s+t)X} = \mathrm{e}^{sX}\,\mathrm{e}^{tX}$. As a result, γ_X is a one parameter subgroup of $\mathrm{GL}(n, \mathbb{C})$. Thus by Proposition 4.1.9, there exists a unique matrix Y such that $\gamma_X(t) = \mathrm{e}^{tY}$ for all t.

Now, the claim is that $\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$, defined by

$$\dot{\rho}(X) := Y \quad \text{such that} \quad \rho(\mathrm{e}^{tX}) = \mathrm{e}^{tY},$$

is the unique Lie algebra matrix representation in mind.

First, $\dot{\rho}$ is well-defined by Proposition 4.1.9. Next, $\gamma_X(1) = \rho(\mathrm{e}^X)$. Thus by the definition of $\dot{\rho}(X)$,

$$\rho(\mathrm{e}^X) = \mathrm{e}^{\dot{\rho}(X)}.$$

Furthermore,

$$\dot{\rho}(X) = \frac{d}{dt}\left(\mathrm{e}^{t\dot{\rho}(X)}\right)|_{t=0} = \frac{d}{dt}\left(\rho(\mathrm{e}^{tX})\right)\Big|_{t=0}.$$

Now one just needs to confirm that $\dot{\rho}$ is a Lie algebra matrix representation of $\mathfrak{g}$. To start, $\dot{\rho}$ is linear. Indeed, let $b \in \mathbb{R}$, then $\gamma_X(tb) = \mathrm{e}^{tb\dot{\rho}(X)}$. By comparing this to $\gamma_X(t) = \mathrm{e}^{t\dot{\rho}(X)}$,

$$\dot{\rho}(bX) = b\dot{\rho}(X).$$

Now let $Z \in \mathfrak{g}$. Then, by use of Lie's Product Formula and the continuity of ρ,

$$\begin{aligned}
\rho(\mathrm{e}^{t(X+Z)}) &= \rho(\mathrm{e}^{tX+tZ}) \\
&= \rho\left(\lim_{m\to\infty}\left(\mathrm{e}^{\frac{tX}{m}}\,\mathrm{e}^{\frac{tZ}{m}}\right)^m\right) \\
&= \lim_{m\to\infty}\left(\rho(\mathrm{e}^{\frac{tX}{m}})\rho(\mathrm{e}^{\frac{tZ}{m}})\right)^m \\
&= \lim_{m\to\infty}\left(\mathrm{e}^{\frac{t\dot{\rho}(X)}{m}}\,\mathrm{e}^{\frac{t\dot{\rho}(Z)}{m}}\right)^m \\
&= \mathrm{e}^{t\dot{\rho}(X)+t\dot{\rho}(Z)}.
\end{aligned}$$

Thus $\rho(\mathrm{e}^{t(X+Z)}) = \mathrm{e}^{t(\dot{\rho}(X)+\dot{\rho}(Z))}$. But $\dot{\rho}(X+Z)$ is uniquely determined from $X+Z$. Therefore

$$\dot{\rho}(X+Z) = \dot{\rho}(X) + \dot{\rho}(Z).$$

Finally, $\dot{\rho}(XZ - ZX) = \dot{\rho}(X)\dot{\rho}(Z) - \dot{\rho}(Z)\dot{\rho}(X)$. This is shown by the following. Let $A \in G$. Then,

$$\mathrm{e}^{t\dot{\rho}(AZA^{-1})} = \rho(\mathrm{e}^{tAZA^{-1}}) = \rho(A\,\mathrm{e}^{tZ}\,A^{-1}) = \rho(A)\rho(\mathrm{e}^{tZ})\rho(A)^{-1}.$$

However,

$$\rho(A)\rho(\mathrm{e}^{tZ})\rho(A)^{-1} = \rho(A)\,\mathrm{e}^{t\dot{\rho}(Z)}\,\rho(A)^{-1} = \mathrm{e}^{t\rho(A)\dot{\rho}(Z)\rho(A)^{-1}}.$$

Hence, $\dot{\rho}(AZA^{-1}) = \rho(A)\dot{\rho}(Z)\rho(A)^{-1}$, by uniqueness. Now using the identity

$$\frac{d}{dt}\left(\mathrm{e}^{tX}\,Y\,\mathrm{e}^{-tX}\right)\Big|_{t=0} = XY - YX = [X,Y],$$

one has

$$\begin{aligned}
[\dot{\rho}(X)\dot{\rho}(Z)] &= \frac{d}{dt}\left(\mathrm{e}^{t\dot{\rho}(X)}\,\dot{\rho}(Z)\,\mathrm{e}^{-t\dot{\rho}(X)}\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\rho(\mathrm{e}^{tX})\dot{\rho}(Z)\rho(\mathrm{e}^{-tX})\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\dot{\rho}(\mathrm{e}^{tX}\,Z\,\mathrm{e}^{-tX})\right)\Big|_{t=0} \\
&= \lim_{t\to 0}\frac{\dot{\rho}(\mathrm{e}^{tX}\,Z\,\mathrm{e}^{-tX}) - \dot{\rho}(Z)}{t} \\
&= \dot{\rho}\left(\lim_{t\to 0}\frac{\mathrm{e}^{tX}\,Z\,\mathrm{e}^{-tX} - Z}{t}\right) \\
&= \dot{\rho}([X,Z]).
\end{aligned}$$

□

Corollary 4.2.5. *Let $\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(V)$ be induced by $\rho : G \to \mathrm{GL}(V)$. Then*

$$\dot{\rho}(AXA^{-1}) = \rho(A)\dot{\rho}(X)\rho(A^{-1})$$

for all $X \in \mathfrak{g}$, and $A \in G$.

Remark. At this point, one can't appeal to $\dot{\rho}$ being the differential of a smooth map between manifolds, since it has yet to be established that ρ, being a Lie group representation, is a smooth map. (See Corollary 4.4.4)

Complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$ are to be put into one to one correspondence with representations of $\mathrm{SU}(n)$. Therefore when considering matrix Lie groups $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, only complex analytic representations will be of interest.

4.3 The complexification of $\mathfrak{su}(n)$.

Complexification is the first step towards connecting complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$ with representations of $\mathrm{SU}(n)$. This correspondence ultimately rests on the fact that the complexification of the real Lie algebra $\mathfrak{su}(n)$ is isomorphic, as a complex Lie algebra, to $\mathfrak{sl}(n, \mathbb{C})$.

Definition 4.3.1. Let $\mathfrak{g}$ be a real Lie algebra. Then the *complexification of* $\mathfrak{g}$, denoted $\mathfrak{g}_{\mathbb{C}}$, is the complex Lie algebra

$$\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}.$$

Scalar multiplication by i is obtained by setting

$$i(X + iY) := -Y + iX,$$

and the Lie bracket is obtained through the assignment

$$[X_1 + iY_1, X_2 + iY_2] := [X_1 X_2] - [Y_1 Y_2] + i\left([X_1 Y_2] + [Y_1 X_2]\right).$$

Remark. The notation $X \otimes (u + iv)$ is replaced by $uX + viX$ due to the more natural appearance of the latter.

To show that the complexification of $\mathfrak{su}(n)$ is isomorphic to $\mathfrak{sl}(n, \mathbb{C})$, the following basis for $\mathfrak{sl}(n, \mathbb{C})$ will be utilized.

For $i, j \in [n]$, let E_{ij} denote the matrix such that there is a 1 in the (i, j)th position and a 0 elsewhere. For $i \in [n-1]$, set $H_i := E_{ii} - E_{i+1i+1}$. Then the basis of consideration is

$$\mathcal{E} = \{E_{ij} \mid 1 \leq i \neq j \leq n\} \cup \{H_i \mid i \in [n-1]\}. \tag{4.3.1}$$

Theorem 4.3.2. *As complex Lie algebras,*

$$\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n, \mathbb{C}).$$

Proof. First, one will need a basis of $\mathfrak{su}(n)$ in which to work. Suppose $X \in \mathrm{SU}(n)$. Then $X^\dagger = -X$, which implies that

$$X = \begin{bmatrix} ih_1 & x_{12} & \cdots & x_{1n} \\ -\overline{x}_{12} & ih_2 & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{x}_{1n} & -\overline{x}_{2n} & \cdots & ih_n \end{bmatrix},$$

for some collection $x_{i,j} \in \mathbb{C}$, and collection $h_l \in \mathbb{R}$ such that $h_1 + h_2 + ... + h_n = 0$. This motivates the following selection. For each $1 \leq l < k \leq n$, set

$$X_{lk} := -\frac{i}{2}(E_{kl} + E_{lk}) \quad \text{and} \quad Y_{lk} := \frac{1}{2}(E_{kl} - E_{lk}).$$

In addition, for each $l \in [n-1]$ define

$$T_l := -\frac{i}{2}\left(E_{ll} - E_{l+1l+1}\right).$$

By considering the basis of $\mathfrak{sl}(n, \mathbb{C})$ defined in 4.3.1, for each $l < k$,

$$E_{lk} = -Y_{lk} + iX_{lk} \quad \text{and} \quad E_{kl} = Y_{lk} + iX_{lk},$$

and, for each $l \in [n-1]$,

$$H_l = 2iT_l.$$

Thus the linearly independent set,

$$\{E_{lk} \mid 1 \leq l \neq k \leq n\} \cup \{H_l \mid l \in [n-1\}$$

is contained in the subspace

$$\langle zX \mid X \in \mathfrak{su}(n),\, z \in \mathbb{C}\rangle \subseteq M_n(\mathbb{C}).$$

Finally,

$$\dim\langle zX \mid X \in \mathfrak{su}(n),\, z \in \mathbb{C}\rangle = n^2 - 1 = \dim \mathfrak{sl}(n, \mathbb{C}).$$

Therefore

$$\mathfrak{su}(n)_{\mathbb{C}} \cong \langle zX \mid X \in \mathfrak{su}(n),\, z \in \mathbb{C}\rangle = \mathfrak{sl}(n, \mathbb{C}),$$

□

The true utility of complexification comes from the application of following theorem.

Theorem 4.3.3. *Let $\mathfrak{g}$ be a real Lie algebra. For every Lie algebra representation $p : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$, there exists a unique complex-linear Lie algebra representation $q : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C})$ such that $q|_{\mathfrak{g}} = p$, and*

$$q(X + iY) = p(X) + ip(Y)$$

for all $X, Y \in \mathfrak{g}$.

Furthermore, q is irreducible if and only if p is irreducible; and moreover, two Lie algebra representations p_1 and p_2 are equivalent if and only if their complex extensions q_1 and q_2 are equivalent.

Proof. Define $q : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C})$ by setting

$$q(X + iY) := p(X) + ip(Y)$$

for $X, Y \in \mathfrak{g}$. It is quick to verify that q is a real linear map, so consider

$$\begin{aligned} q(i\,(X + iY)) &= q(-Y + iX) \\ &= p(-Y) + ip(X) \\ &= -p(Y) + ip(X) \\ &= i\,(p(X) + ip(Y))\,. \\ &= iq(X + iY). \end{aligned}$$

Consequently, q is a complex linear transformation.

Now let $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$. Then,

$$\begin{aligned} q([X_1 + iY_1, X_2 + iY_2]) &= q\left([X_1, X_2] - [Y_1, Y_2] + i\left([X_1, Y_2] + [Y_1, X_2]\right)\right). \\ &= p\left([X_1, X_2] - [Y_1, Y_2]\right) + ip\left([X_1, Y_2] + [Y_1, X_2]\right) \\ &= [p(X_1), p(X_2)] - [p(Y_1), p(Y_2)] + i\left([p(X_1), p(Y_2)] + [p(Y_1), p(X_2)]\right) \\ &= [p(X_1) + ip(Y_1), p(X_2) + ip(Y_2)] \\ &= [q(X_1 + iY_1), q(X_2 + iY_2)]. \end{aligned}$$

Therefore, q is a complex linear Lie algebra homomorphism.

By computing $q(X + iY)$ with $Y = 0$, it's clear that $q|_{\mathfrak{g}} = p$. To see that q is unique, consider the following. Suppose $r : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(n, \mathbb{C})$ is another Lie algebra homomorphism such that $r|_{\mathfrak{g}} = p$. Then $r(X) = p(X)$ for all $X \in \mathfrak{g}$. Furthermore, r is complex linear. Thus

$$r(X + iY) = r(X) + ir(Y) = p(X) + ip(Y)$$

for $X, Y \in \mathfrak{g}$. Therefore $r = q$.

Suppose p is irreducible, and W is a $\mathfrak{g}_{\mathbb{C}}-$submodule of V. But $q|_{\mathfrak{g}} = p$. Thus W is a $\mathfrak{g}-$submodule. As a result, W must be trivial. Therefore q is irreducible.

Conversely, suppose that q is irreducible, and W is a $\mathfrak{g}-$submodule. Let $w \in W$, and $X, Y \in \mathfrak{g}$. Then

$$q(X)w = (p(X) + ip(Y))\,w = p(X)w + ip(Y)w.$$

By hypothesis, $p(X)w \in W$ and $ip(Y)w \in W$. Thus

$$q(X)w = p(X)w + ip(Y)w \in W.$$

Consequently W is $\mathfrak{g}_{\mathbb{C}}-$submodule of V. Hence W is trivial, and therefore p is irreducible.

Finally, suppose p_1 and p_2 are two equivalent Lie algebra representations for $\mathfrak{g}$, and let

$\phi : \mathbb{C}^n \to \mathbb{C}^n$ be a corresponding $\mathfrak{g}-$ isomorphism. Let $X, Y \in \mathfrak{g}$, and $v \in \mathbb{C}^n$. Then

$$\begin{aligned}
q_1(X+iY)(\phi(v)) &= (p_1(X) + ip_1(Y))\,(\phi(v)) \\
&= p_1(X)(\phi(v)) + i\,(p_1(Y)(\phi(v))) \\
&= \phi(p_2(X)(v)) + i\,(\phi(p_2(Y)(v))) \\
&= \phi(p_2(X)(v)) + \phi(ip_2(Y)(v)) \\
&= \phi\,(p_2(X)(v) + ip_2(Y)(v)) \\
&= \phi\,(q_2(X+iY)(v))\,.
\end{aligned}$$

Thus q_1 and q_2 are equivalent as well. The reverse implication is trival, and therefore the proof is complete. □

Theorem 4.3.3 will have profound influences in Section 4.4.

4.4 The correspondence between $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$.

Two main goals are present for this concluding section. The first is to establish that matrix Lie groups are smooth manifolds, with the additional property that complex matrix Lie groups are complex manifolds. This establishment will allow for justification that Lie group representations, as currently defined, are already smooth, and that complex analytic Lie group representations give rise to, and result from the existence of a complex linear Lie algebra representations. The second is to finalize the one to one correspondence between the complex analytic Lie group representations of $\mathrm{SL}(n, \mathbb{C})$ and the Lie group representations of $\mathrm{SU}(n)$.

Lemma 4.4.1. *Let G be a matrix Lie group. Suppose there exists a sequence $(A_k)_{k\in\mathbb{N}}$ in $G \setminus \{I\}$, such that $\log A_k$ is defined for all k, and*

$$\lim_{k\to\infty} A_k = I.$$

If, for some $X \in M_n(\mathbb{C})$,

$$\lim_{k\to\infty} \frac{\log A_k}{||\log A_k||} = X,$$

then $X \in \mathfrak{g}$.

Proof. First note that if $(a_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, then, for all $t \in \mathbb{R}$, one can choose integers m_k such that

$$\lim_{k\to\infty} m_k a_k = t.$$

Now, $A_k^{m_k} \in G$ for all k, and

$$\begin{aligned}
A_k^{m_k} &= \mathrm{e}^{m_k \log A_k} \\
&= \mathrm{e}^{(m_k ||\log A_k||)\frac{\log A_k}{||\log A_k||}}\,.
\end{aligned}$$

Thus

$$\lim_{k\to\infty}\left((m_k||\log A_k||)\frac{\log A_k}{||\log A_k||}\right) = tX.$$

Consequently

$$\begin{aligned} e^{tX} &= \lim_{k\to\infty} e^{(m_k||\log A_k||)\frac{\log A_k}{||\log A_k||}} \\ &= \lim_{k\to\infty} A_k^{m_k}. \end{aligned}$$

As a result, $e^{tX} \in G$ since G is closed. Therefore $X \in \mathfrak{g}$. $\square$

The following theorem shows that, for each matrix Lie Group G, the matrix exponential maps an open neighborhood about $0 \in \mathfrak{g}$, homeomorphically onto an open neighborhood about $I \in G$.

Theorem 4.4.2. *For all* $0 < \varepsilon < \ln 2$, *let*

$$N_\varepsilon = \{X \in M_n(\mathbb{C}) \mid ||X|| < \varepsilon\}$$

denote the $\varepsilon-$*ball about the zero matrix. The*

$$e(N_\varepsilon) \subseteq \mathrm{GL}(n,\mathbb{C}),$$

is an open neighborhood about $I \in \mathrm{GL}(n,\mathbb{C})$ *homeomorphic to* N_ε, *via* $e|_{N_\varepsilon}$.

Furthermore, for each matrix Lie group G, *there exists an* $0 < \epsilon < \ln 2$ *such that*

$$e(N_\epsilon \cap \mathfrak{g}) = e(N_\varepsilon) \cap G,$$

where $e|_{N_\epsilon\cap\mathfrak{g}}$ *is a homeomorphism.*

Proof. The first assertion of theorem is immediate from Proposition 4.1.5. So, consider a matrix Lie group G with its Lie algebra $\mathfrak{g}$. By the definition of $\mathfrak{g}$,

$$e(N_\epsilon \cap \mathfrak{g}) \subseteq e(N_\varepsilon) \cap G,$$

whenever $0 < \epsilon < \ln 2$. What needs to be shown is that, there exists an $\epsilon \in (0, \ln 2)$, such that, for each $A \in e(N_\varepsilon)$,

$$\log A \in \mathfrak{g}$$

whenever $A \in G$. The argument proceeds using contradiction.

Suppose not. Then there exists a sequence $(A_k)_{k\in\mathbb{N}}$ in $G \setminus \{I\}$ such that

$$\lim_{k\to\infty} A_k = I,$$

yet, for each k, $\log A_k \notin \mathfrak{g}$. Now consider $M_n(\mathbb{C})$ as $\mathbb{C}^{n^2}$ with the standard inner product, $\langle\cdot,\cdot\rangle$. Then $\log A_k \notin \mathfrak{g}$ implies there exists an $X_k \in \mathfrak{g}$, and $Y_k \in \mathfrak{g}^\perp \setminus \{0\}$ such that

$$A_k = e^{X_k} e^{Y_k}$$

with both X_k and Y_k tending to 0 as k goes to ∞. Indeed,

$$M_n(\mathbb{C}) = \mathfrak{g} \oplus \mathfrak{g}^\perp.$$

So define $\varphi : M_n(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ by

$$\varphi(B) := \mathrm{e}^X \mathrm{e}^Y,$$

where $X \in \mathfrak{g}$, and $Y \in \mathfrak{g}^\perp$ such that $B = X + Y$. Note that φ is well defined, and has a derivative equal to the identity at 0. Consequently, by the inverse function theorem, φ has a continuous local inverse, defined in some neighborhood of I.

For $A_k = \mathrm{e}^{X_k} \mathrm{e}^{Y_k}$, $Y_k \neq 0$. Otherwise, $A_k = \mathrm{e}^{X_k}$, and hence $\log A_k = X_k \in \mathfrak{g}$. Now, set

$$B_k = \mathrm{e}^{-X_k} A_k = \mathrm{e}^{Y_k}.$$

Then

$$\lim_{k\to\infty} B_k = I.$$

Furthermore, since $\{Y \in \mathfrak{g}^\perp \mid ||Y|| = 1\}$ is compact, find a subsequence $(Y_{k_j})_{j\in\mathbb{N}}$ such that, for some $Y_L \in \mathfrak{g}^\perp$ with $||Y_L|| = 1$,

$$\lim_{j\to\infty} \left(\frac{Y_{k_j}}{||Y_{k_j}||} \right) = Y_L.$$

By Lemma 4.4.1,

$$Y_L \in \mathfrak{g}.$$

However, this means $Y_L = 0$ since $Y_L \in \mathfrak{g} \cap \mathfrak{g}^\perp$. This contradicts the fact that $||Y_L|| = 1$. Therefore there exists an $\epsilon \in (0, \ln 2)$, such that

$$\mathrm{e}(N_\varepsilon) \cap G \subseteq \mathrm{e}(N_\epsilon \cap \mathfrak{g}).$$

Finally, considering Proposition 4.1.5,

$$\mathrm{e}\,|_{N_\epsilon \cap \mathfrak{g}} : N_\epsilon \cap \mathfrak{g} \to \mathrm{e}(N_\varepsilon) \cap G$$

is a continuous bijection with continuous inverse

$$\log|_{\mathrm{e}(N_\varepsilon)\cap G} : \mathrm{e}(N_\varepsilon) \cap G \to N_\epsilon \cap \mathfrak{g}.$$

Therefore $\mathrm{e}\,|_{N_\epsilon \cap \mathfrak{g}}$ is a homeomorphism, and the neighborhoods $N_\epsilon \cap \mathfrak{g}$ and $\mathrm{e}(N_\varepsilon) \cap G$ are homeomorphic. □

This has far reaching consequences. The first is that matrix Lie groups are smooth manifolds with complex matrix Lie groups being complex manifolds. The following corollary addresses the case for $\mathrm{SU}(n)$ and $\mathrm{SL}(n, \mathbb{C})$.

Corollary 4.4.3. $\mathrm{SU}(n)$ *is a smooth manifold of (real) dimension* $n^2 - 1$. $\mathrm{SL}(n, \mathbb{C})$ *is a complex manifold with complex dimension* $n^2 - 1$.

Proof. Pick $0 < \epsilon < \ln 2$ such that

$$\mathrm{e}(N_\epsilon \cap \mathfrak{su}(n)) = \mathrm{e}(N_\epsilon) \cap \mathrm{SU}(n)$$

with $\mathrm{e}\,|_{N_\epsilon \cap \mathfrak{su}(n)}$ being the homeomorphism. Set $N_0 = N_\epsilon \cap \mathfrak{su}(n)$, and $N_I = \mathrm{e}(N_\epsilon) \cap \mathrm{SU}(n)$. Since they are defined in terms of power series, $\mathrm{e}\,|_{N_0} : N_0 \to N_I$ is an analytic map, and its local inverse $\log|_{N_I} : N_I \to N_0$ is analytic as well. Also recall, for each $A \in \mathrm{SU}(n)$, the map

$$L_A : B \to AB,$$

is smooth on $M_n(\mathbb{C})$.Set $N_A = AN_I$, and define $\varphi_A : N_A \to N_0$ by

$$\varphi_A(B) = \log(A^{-1}B).$$

Then the following collection forms a smooth atlas for $\mathrm{SU}(n)$

$$\{(N_A, \varphi_A) \mid A \in \mathrm{SU}(n)\}.$$

Therefore $\mathrm{SU}(n)$ is a smooth manifold of (real) dimension $\dim \mathfrak{su}(n) = n^2 - 1$.

Note: Technically, one should have chosen a basis of $\mathfrak{su}(n)$ *to identify* $\mathfrak{su}(n)$ *with* $\mathbb{R}^{n^2-1}$ *and* N_0 *with some open subset of* $\mathbb{R}^{n^2-1}$. *Then one would have a legitimate smooth atlas of* $\mathrm{SU}(n)$. *However, implementing this into the previous argument would further complicate things through a more cumbersome system notation.*

Consider $\mathrm{SL}(n, \mathbb{C})$. Its Lie algebra, $\mathfrak{sl}(n, \mathbb{C})$, is complex. Thus the same argument used in the case for $\mathrm{SU}(n)$ shows that $\mathrm{SL}(n, \mathbb{C})$ is a smooth manifold of (real) dimension $2(n^2 - 1)$. However, consider the smooth atlas

$$\{(N_A, \varphi_A) \mid A \in \mathrm{SL}(n, \mathbb{C})\},$$

with the transition functions

$$\{\varphi_A \circ \varphi_B^{-1}|_{\varphi_B(N_A \cap N_B)} : \varphi_B(N_A \cap N_B) \to \varphi_A(N_A \cap N_B) \mid N_A \cap N_B \neq \emptyset\}.$$

First, $\varphi_B(N_A \cap N_B)$ and $\varphi_A(N_A \cap N_B)$ are open subsets of $\mathbb{C}^{n^2-1}$. Second,

$$L_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

is complex analytic for each $A \in M_n(\mathbb{C})$. Thus

$$\varphi_A \circ \varphi_B^{-1}|_{\varphi_B(N_A \cap N_B)} = \log \circ L_{A^{-1}B} \circ \mathrm{e}\,|_{\varphi_B(N_A \cap N_B)}$$

is a complex analytic map for all $A, B \in \mathrm{SL}(n, \mathbb{C})$ such that $N_A \cap N_B \neq \emptyset$. Hence

$$\{(N_A, \varphi_A) \mid A \in \mathrm{SL}(n, \mathbb{C})\}$$

is smooth atlas for $\mathrm{SL}(n, \mathbb{C})$ with complex analytic transition functions. Therefore $\mathrm{SL}(n, \mathbb{C})$ is a complex manifold of (complex) dimension $n^2 - 1$. □

Remark. This proof can be adapted to show that any matrix Lie group is a smooth manifold, and additionally, a complex manifold whenever its Lie algebra is complex. In particular, $\mathrm{GL}(n,\mathbb{C})$ is a complex manifold.

A second consequence of Theorem 4.4.2 is the following anticipated result concerning Lie group representations and complex analytic Lie group representations.

Corollary 4.4.4. *If $\rho : G \to \mathrm{GL}(V)$ is a representation for G, then ρ is smooth.*

Moreover, if G is a complex matrix Lie group, then $\dot{\rho} : \mathfrak{g} \to \mathfrak{g}(V)$ is a complex linear Lie algebra representation if and only if ρ is complex analytic.

Proof. Like before, the proof appeals to matrix representations of G.

Let $A \in G$, (φ_A, N_A) be a local chart for A, and $(\phi_{\rho(A)}, N'_{\rho(A)})$ be a local chart for $\rho(A)$ in $\mathrm{GL}(n,\mathbb{C})$, as outlined in Corollary 4.4.3, Suppose, $B \in N_A \cap \rho^{-1}(N'_{\rho(A)})$, then $B = A\,\mathrm{e}^X$ for some $X \in \varphi_A(N_A \cap \rho^{-1}(N'_{\rho(A)}))$. Now,

$$\rho(B) = \rho(A\,\mathrm{e}^X) = \rho(A)\,\mathrm{e}^{\dot{\rho}(X)} .$$

Thus locally,

$$\phi_{\rho(A)} \circ \rho \circ \varphi_A^{-1}|_W = \dot{\rho}|_W,$$

where $W = \varphi_A(N_A \cap \rho^{-1}(N'_{\rho(A)}))$. A real linear map sending W into $\mathbb{C}^{n^2}$ is smooth. Therefore, ρ is smooth on G.

For the second assertion of the corollary, note that if G is a complex matrix Lie group, and

$$\dot{\rho} : \mathfrak{g} \to \mathfrak{g}(n,\mathbb{C})$$

is a complex linear Lie algebra representation, then ρ must be complex analytic. Indeed, from earlier, one has the local expression

$$\phi_{\rho(A)} \circ \rho \circ \varphi_A^{-1}|_W = \dot{\rho}|_W$$

on

$$W = \varphi_A(N_A \cap \rho^{-1}(N'_{\rho(A)})).$$

But now W is open in $\mathbb{C}^{\dim \mathfrak{g}}$, and

$$\phi_{\rho(A)} \circ \rho \circ \varphi_A^{-1}|_W$$

is now given by the complex linear map $\dot{\rho}|_W$. Thus $\phi_{\rho(A)} \circ \rho \circ \varphi_A^{-1}|_W$ is a complex analytic map from W to $\mathbb{C}^{\dim \mathfrak{g}}$. Since this is true for every $A \in G$, ρ must be complex analytic.

Now, let (φ_I, N_I) be a local chart for $I \in G$, and let (ϕ_I, N'_I) be a corresponding local chart for I in $\mathrm{GL}(n,\mathbb{C})$. Suppose that ρ is complex analytic. Let $\{X_l \mid l \in [m]\}$, with $m = \dim \mathfrak{g}$, be a basis for $\mathfrak{g}$, and let $\{E_{lk} \mid 1 \leq l, k \leq n\}$ be the standard (matrix) basis for $\mathfrak{g}(n,\mathbb{C})$. Using these bases, one can get honest coordinates for $N_I \subseteq G$ and $N'_I \subseteq \mathrm{GL}(n,\mathbb{C})$ respectively. To be exact, if

$$A = \varphi_I^{-1}(z_1X_1 + ... + z_mX_m) = \mathrm{e}^{z_1X_1+...+z_mX_m},$$

then $(z_1, ..., z_m) \in \mathbb{C}^m$ are the local coordinates for $A \in N_I$. Likewise, $(z_{lk})_{lk} \in \mathbb{C}^{n^2}$ are local coordinates for $B \in N'_I$, if $B = \phi_I^{-1}(Z) = \mathrm{e}^Z$, where

$$Z = \sum_{l,k=1}^{n} z_{lk} E_{lk} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix}.$$

Now, let $f : U \to \mathbb{C}$ be smooth on $U \subseteq N_I$ such that $I \in U$, and denote $x_l = \mathrm{Re}(z_l)$ with $y_l = \mathrm{Im}(z_l)$. Recall that in the complex setting, one defines the $2m$ partial derivatives relative to $\{X_l \mid l \in [m]\}$ on N_I by the 'pull back' of f to $\varphi_I(U)$. To be exact, let $\hat{f} = f \circ \varphi_I^{-1}|_{\varphi_I(U)}$, then, for each l

$$\frac{\partial}{\partial x_l} f(Y)|_{Y=A} = \frac{d}{dt} \hat{f}\left[\varphi_I(A) + tX_l\right]\Big|_{t=0},$$

and

$$\frac{\partial}{\partial y_l} f(Y)|_{Y=A} = \frac{d}{dt} \hat{f}\left[\varphi_I(A) + t(iX_l)\right]\Big|_{t=0}.$$

For each $l \in [m]$, recall that

$$\frac{\partial}{\partial \bar{z}_l} := \frac{1}{2}\left(\frac{\partial}{\partial x_l} + i\frac{\partial}{\partial y_l}\right).$$

With this in mind, f is also complex analytic (complex manifold sense) on U, whenever $\hat{f}$ is complex analytic (standard sense) on $\varphi_I(U)$. If so, then for all $A \in U$ and $l \in [m]$,

$$\frac{\partial}{\partial \bar{z}_l} f(Y)|_{Y=A} = 0.$$

Furthermore, let $W = \varphi_I(N_I \cap \rho^{-1}(N'_I))$. Let $U = N_I \cap \rho^{-1}(N'_I)$, and let $\Phi : U \to \phi_I(N'_I)$ be given by $\Phi = \phi_I \circ \rho|_U$. Also, for each $l, k \in [n]$, let Φ_{lk} denote the coordinate of Φ relative to E_{lk}. In other words, Φ_{lk} is just the (l,k)th matrix entry of Φ. By definiton of ρ being complex analytic on G, one has that

$$\hat{\Phi}_{lk} : W \to \mathbb{C},$$

is complex analytic, for each $l, k \in [n]$. Thus,

$$\frac{\partial}{\partial \bar{z}_j} \Phi_{lk}(Y)|_{Y=I} = 0,$$

for all $j \in [m]$ and $l, k \in [n]$. Consequently,

$$\frac{\partial}{\partial x_j} \Phi_{lk}(Y)|_{Y=I} = (-i)\frac{\partial}{\partial y_j} \Phi_{lk}(Y)|_{Y=I}.$$

Now, since $\varphi_I(I) = \log(I) = 0$, then

$$\begin{aligned}
\frac{\partial}{\partial x_j}\, \Phi_{lk}(Y)\big|_{Y=I} &= \frac{d}{dt}\, \hat{\Phi}_{lk}\left[tX_j\right]\Big|_{t=0} \\
&= \frac{d}{dt}\, \left(\log(\rho(\mathrm{e}^{tX_j}))\right)_{lk}\Big|_{t=0} \\
&= \frac{d}{dt}\, \left(\log(\mathrm{e}^{t\dot{\rho}(X_j)})\right)_{lk}\Big|_{t=0} \\
&= \frac{d}{dt}\, \left(t\dot{\rho}(X_j)\right)_{lk}\Big|_{t=0} \\
&= (\dot{\rho}(X_j))_{lk}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\frac{\partial}{\partial y_j}\, \Phi_{lk}(Y)\big|_{Y=I} &= \frac{d}{dt}\, \hat{\Phi}_{lk}\left[t(iX_j)\right]\Big|_{t=0} \\
&= \frac{d}{dt}\, \left(t\dot{\rho}(iX_j)\right)_{lk}\Big|_{t=0} \\
&= (\dot{\rho}(iX_j))_{lk}.
\end{aligned}$$

Thus, for all $j \in [m]$ and $l, k \in [n]$, $(\dot{\rho}(X_j))_{lk} = (-i)(\dot{\rho}(iX_j))_{lk}$ and hence

$$(\dot{\rho}(iX_j))_{lk} = i(\dot{\rho}(X_j))_{lk} = (i\dot{\rho}(X_j))_{lk}.$$

Therefore, $\dot{\rho} : \mathfrak{g} \to \mathfrak{g}(n, \mathbb{C})$ is a complex linear Lie algebra representation, since

$$\dot{\rho}(iX_j) = i\dot{\rho}(X_j),$$

for all $j \in [m]$, and $\{X_j \mid j \in [m]\}$ is a basis for $\mathfrak{g}$. □

Next is a verification that Lie algebras of matrix Lie groups are in fact tangent spaces of manifold theory.

Corollary 4.4.5. *Let G be a matrix Lie group. Then, $\mathfrak{g}$ is the tangent space to G at the identity.*

Proof. There are more than one equivalent characterizations of the tangent space at point on a smooth manifold. This proof defines the tangent space at $I \in G$ to be 'the set of equivalence classes of smooth curves passing through I ' To be exact, two smooth curves at $I \in G$ are equivalent, denoted

$$\gamma \sim \Gamma,$$

if $\gamma(0) = \Gamma(0) = I$, and

$$\frac{d}{dt}\, (\gamma(t))\big|_{t=0} = \frac{d}{dt}\, (\Gamma(t))\big|_{t=0}.$$

Naturally, one denotes the equivalence class $[\gamma]$ by the common derivative of the curves.

To start, if $X \in \mathfrak{g}$, then the map $\exp_X(t) = \mathrm{e}^{tX}$ clearly represents an equivalence class, i.e. $\gamma \sim \exp_X$, whenever $\gamma(0) = I$, and $\frac{d}{dt}\left(\gamma(t)\right)|_{t=0} = X$. So X itself denotes this equivalence class.

Conversely, suppose that γ is a smooth curve defined on some interval about zero such that $\gamma(0) = I$ and $\frac{d}{dt}\left(\gamma(t)\right)|_{t=0} = X$. Then, from Theorem 4.4.2, let $N_0 \subseteq \mathfrak{g}$ and $N_I \subseteq G$ be two homeomorphic open neighborhoods about 0 and I respectively. Now by continuity, find a positive ε such that $\gamma((-\varepsilon, \varepsilon)) \subseteq N_I$. Then $\log \gamma(t) \in N_0$, for all $t \in (-\varepsilon, \varepsilon)$. From here, note that if a smooth curve Γ in G satisfies

$$\Gamma(0) = I,$$

then

$$\frac{d}{dt}\left(\log \Gamma(t)\right)|_{t=0} = \frac{d}{dt}\left(\Gamma(t)\right)|_{t=0}.$$

So for this case, one has $\frac{d}{dt}\left(\log \gamma(t)\right)|_{t=0} = X$. Thus,

$$\lim_{t \to 0} \frac{\log \gamma(t) - 0}{t} = X.$$

In other words, X is the limit of matrices in $\mathfrak{g}$. Hence, $X \in \mathfrak{g}$. Therefore, the equivalence class containing γ is given by an element of $\mathfrak{g}$. With this the proof is complete. $\square$

Now, the following two results, Corollary 4.4.6 and Lemma 4.4.7, are appropriate since connectedness is crucial in passing properties, like irreducibility of a Lie group representation down to the induced Lie algebra representation. In fact, this will soon be addressed in Theorem 4.4.8.

Corollary 4.4.6. *Let G be a connected matrix Lie group, then there exist some open neighborhood $N \subseteq \mathfrak{g}$ about 0, such that*

$$G = \langle \mathrm{e}^X \mid X \in N \rangle.$$

In other words, for all $A \in G$, there exists a collection $\{X_i \mid i \in [m]\} \subseteq \mathfrak{g}$, such that

$$A = \mathrm{e}^{X_1}\, \mathrm{e}^{X_2} \ldots \mathrm{e}^{X_m}.$$

Proof. Connect A and I with a continuous path $\gamma : [0,1] \to G$, where $\gamma(0) = I$ and $\gamma(1) = A$. Then, find an partition of the path

$$\{A_k \mid k \in \{0, \ldots K\}\}$$

so fine, one has that

$$A_l (A_{l-1})^{-1} \in \mathrm{e}(N)$$

for all $l \in [K]$. As a result, one can pick a $X_l \in N$, such that

$$\mathrm{e}^{X_l} = A_l (A_{l-1})^{-1}.$$

Therefore,

$$A = A_K(A_{K-1})^{-1}A_{K-1}....(A_1)^{-1}A_1A_0 = \mathrm{e}^{X_K}\,\mathrm{e}^{X_{K-1}}\,...\,\mathrm{e}^{X_1},$$

where $A_0 = I$ and $A = A_K$. $\square$

Lemma 4.4.7. *Let $\rho : G \to \mathrm{GL}(V)$ be a Lie group representation, and $\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced Lie algebra representation. If the matrix Lie group G is connected, then a subspace $W \leq V$ is a G- submodule if and only if W is a $\mathfrak{g}-$submodule.*

Proof. Let $w \in W$, suppose W is a $G-$submodule and let $X \in \mathfrak{g}$. Then, by Proposition 4.2.4,

$$\dot{\rho}(X)w = \lim_{t\to 0}\frac{\rho(\mathrm{e}^{tX})w - w}{t}.$$

For each $t \in \mathbb{R}$,

$$\frac{\rho(\mathrm{e}^{tX})w - w}{t} \in W$$

since is W is assumed to be a $G-$submodule. However, subspaces are closed. Therefore $\dot{\rho}(X)w \in W$.

Conversely, suppose that W is a $\mathfrak{g}-$submodule, and let $A \in G$. G is connected, so using Corollary 4.4.6, find $\{X_l \mid l \in [K]\}$ such that

$$A = \mathrm{e}^{X_K}\,\mathrm{e}^{X_{K-1}}\,...\,\mathrm{e}^{X_1}.$$

By Proposition 4.2.4, $\rho(\mathrm{e}^{X_l}) = \mathrm{e}^{\dot{\rho}(X_l)}$ for each l. Thus

$$\rho(A)w = \mathrm{e}^{\dot{\rho}(X_K)}\,\mathrm{e}^{\dot{\rho}(X_{K-1})}\,...\,\mathrm{e}^{\dot{\rho}(X_1)}\,w.$$

Once it is shown that

$$\mathrm{e}^{\dot{\rho}(X)}\,w \in W$$

for any $X \in \mathfrak{g}$, the proof will be complete.

Using the definition of the matrix exponetial, write

$$\mathrm{e}^{\dot{\rho}(X)}\,w = \lim_{m\to\infty}\left(\sum_{k=0}^{m}\frac{\dot{\rho}(X)}{k!}w\right).$$

Under the assumption that W is a $\mathfrak{g}$-submodule,

$$\sum_{k=0}^{m}\frac{\dot{\rho}(X)}{k!}w \in W$$

for each $m \in \mathbb{N}$. Therefore since W is closed,

$$\mathrm{e}^{\dot{\rho}(X)}\,w \in W.$$

$\square$

Theorem 4.4.8. *Let $\rho : G \to \mathrm{GL}(V)$ be a Lie group representation, and $\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(V)$ be the induced Lie algebra representation. Suppose G is a connected matrix Lie group. Then ρ is irreducible if and only if $\dot{\rho}$ is irreducible.*

Furthermore, two Lie group representations ρ and ϱ for G are equivalent if and only if their induced Lie algebra representations $\dot{\rho}$ and $\dot{\varrho}$ are equivalent.

Proof. Suppose ρ is irreducible. Then the only $G-$submodules are the trivial ones. Therefore since G is connected, by Lemma 4.4.7, the only $\mathfrak{g}-$submodules are trivial. Consequently, $\dot{\rho}$ is irreducible. The converse is shown by the same argument.

Let $\rho : G \to \mathrm{GL}(V)$, and $\varrho : G \to \mathrm{GL}(W)$ be representations for G. Suppose there exists an invertible linear transformation $\phi : V \to W$ such that $\phi(\rho(A)v) = \varrho(A)\phi(v)$ for all $A \in G$ and $v \in V$. Then for $X \in \mathfrak{g}$

$$\phi\left(\frac{\rho(\mathrm{e}^{tX}) - I}{t} v\right) = \frac{\phi(\rho(\mathrm{e}^{tX})v) - \phi(v)}{t} = \frac{\varrho(\mathrm{e}^{tX})\phi(v) - \phi(v)}{t} = \left(\frac{\varrho(\mathrm{e}^{tX}) - I}{t}\right)\phi(v),$$

for all $t \in \mathbb{R}$. Hence, by the continuity of the constant linear transformation ϕ,

$$\phi \circ \left(\frac{d}{dt}\left(\rho(\mathrm{e}^{tX})\right)\Big|_{t=0}\right) = \left(\frac{d}{dt}\left(\varrho(\mathrm{e}^{tX})\right)\Big|_{t=0}\right) \circ \phi.$$

Therefore, $\dot{\rho}$ and $\dot{\varrho}$ are equivalent.

Conversely, suppose that $\dot{\rho}$ and $\dot{\varrho}$ are equivalent via $\phi : V \to W$. Then, $\phi(\dot{\rho}(X)v) = \dot{\varrho}(X)\phi(v)$, for all $X \in \mathfrak{g}$ and $v \in V$. This implies that, for each $m \in \mathbb{N}$,

$$\phi\left(\sum_{k=0}^{m} \frac{\dot{\rho}(X)^k v}{k!}\right) = \sum_{k=0}^{m} \frac{\phi(\dot{\rho}(X)^k v)}{k!} = \sum_{k=0}^{m} \frac{\dot{\varrho}(X)^k \phi(v)}{k!}.$$

Indeed,

$$\phi(\dot{\rho}(X)^k v) = \dot{\varrho}(X)^k \phi(v)$$

for each $k \in \mathbb{N}$. (Use induction to see this.) Thus

$$\phi(\mathrm{e}^{\dot{\rho}(X)}\, v) = \mathrm{e}^{\dot{\varrho}(X)}\, \phi(v).$$

Induction will also show that

$$\phi(\mathrm{e}^{\dot{\rho}(X_1)}\, \mathrm{e}^{\dot{\rho}(X_2)} \ldots \mathrm{e}^{\dot{\rho}(X_k)}\, v) = \mathrm{e}^{\dot{\varrho}(X_1)}\, \mathrm{e}^{\dot{\varrho}(X_2)} \ldots \mathrm{e}^{\dot{\varrho}(X_k)}\, \phi(v)$$

for all $k \in \mathbb{N}$. Since G is connected,

$$\begin{aligned}
\phi \circ \rho(A) &= \phi \circ \rho\left(e^{X_K} e^{X_{K-1}} \ldots e^{X_1}\right) \\
&= \phi \circ \left(e^{\dot{\rho}(X_1)} e^{\dot{\rho}(X_2)} \ldots e^{\dot{\rho}(X_k)}\right) \\
&= \left(e^{\dot{\varrho}(X_1)} e^{\dot{\varrho}(X_2)} \ldots e^{\dot{\varrho}(X_k)}\right) \circ \phi \\
&= \varrho\left(e^{X_1} e^{X_2} \ldots e^{X_k}\right) \circ \phi \\
&= \varrho(A) \circ \phi,
\end{aligned}$$

where the collection $\{X_l \in \mathfrak{g} \mid l \in [K]\}$ was chosen, such that $A = e^{X_K} e^{X_{K-1}} \ldots e^{X_1}$. Therefore ρ and ϱ are equivalent. □

The last remaining technicality concerns lifting a Lie algebra representation from the Lie algebra to a unique Lie group representation on the matrix Lie group. When the matrix Lie group is simply connected, in addition to being connected, the following theorem, in a sense, serves as a converse to Theorem 4.4.8.

Theorem 4.4.9. *Let G be a connected matrix Lie group, and $p : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ be a Lie algebra representation. If G is simply-connected, then there exists a unique Lie group representation $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ such that $\rho(e^X) = e^{p(X)}$ for all $X \in \mathfrak{g}$.*

In [1], pg 76-79, is a nice proof of Theorem 4.4.9 using the Baker-Campbell-Hausdorff formula. Unfortunately, the proof is quite involved. It will be more practical (and productive) to just provide an outline of the proof. Use of the Baker-Campbell-Hausdorff formula plays a key part in the proof. Therefore it will be beneficial to present the statement of the formula before the outline of Theorem 4.4.9.

Recall that $\mathrm{Hom}_{\mathbb{C}}(V, V)$ can be identified with $M_n(\mathbb{C})$ through the choice of an arbitrary basis $\mathcal{B}$. Thus, if $T \in \mathrm{Hom}_{\mathbb{C}}(V, V)$, then define the norm

$$||T|| := ||[T]_{\mathcal{B}}||,$$

where $[T]_{\mathcal{B}}$ is the matrix of T relative to $\mathcal{B}$.

Lemma 4.4.10. *The complex function*

$$g(z) = \frac{\log(z)}{1 - \frac{1}{z}},$$

is defined, and is analytic on the open disk $\{z \mid |z - 1| < 1\}$. Hence, for some collection of coefficients $\{a_m\}$,

$$g(w) = \sum_{k=0}^{\infty} a_k (w - 1)^k$$

for all $w \in \{z \mid |z - 1| < 1\}$.

Furthermore, using the same collection of coefficients $\{a_m\}$, *the operator function*

$$g(T) := \sum_{k=0}^{\infty} a_k (T - I)^k$$

is defined for all $T \in \mathrm{Hom}_{\mathbb{C}}(V, V)$ *such that* $||T - I|| < 1$, *where* I *denotes* id_V.

Define $\mathrm{ad}_X(Y) := [X, Y]$ for $X, Y \in M_n(\mathbb{C})$. Then

$$\mathrm{e}^{\mathrm{ad}_X}(Y) = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{6}[X, [X, [XY]]] + ...$$

With this is the statement of the Baker-Campbell-Hausdorff formula.

Theorem 4.4.11. *For all* $X, Y \in M_n(\mathbb{C})$ *with* $||X||$ *and* $||Y||$ *sufficiently small so that* $||\,\mathrm{e}^{\mathrm{ad}_X}\mathrm{e}^{t\mathrm{ad}_Y} - I|| < 1$ *for all* $t \in [0, 1]$,

$$\log(\mathrm{e}^X \mathrm{e}^Y) = X + \int_0^1 g\left(\mathrm{e}^{\mathrm{ad}_X}\mathrm{e}^{t\mathrm{ad}_Y}\right)(Y)dt.$$

Remark. Here are the first few terms of the Baker-Campbell-Hausdorff formula

$$\log(\mathrm{e}^X \mathrm{e}^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \text{higher order terms},$$

where the higher order terms only involve X, Y, Lie brackets of X and Y, Lie brackets of Lie brackets of X and Y, etc. What is important about this formula is that $\log(\mathrm{e}^X \mathrm{e}^Y)$ can be expressed completely in terms of brackets. Consequently one has the following result.

Corollary 4.4.12. *Let* G *be a matrix Lie group. Suppose* $p : \mathfrak{g} \to \mathfrak{gl}(V)$ *is a Lie algebra representation. Then for all* $X, Y \in \mathfrak{g}$ *with* $||X||$ *and* $||Y||$ *sufficiently small,*

$$\log(\mathrm{e}^X \mathrm{e}^Y) \in \mathfrak{g},$$

and

$$p\left(\log(\mathrm{e}^X \mathrm{e}^Y)\right) = \log(\mathrm{e}^{p(X)}\mathrm{e}^{p(Y)}).$$

Proof. Let $X, Y \in \mathfrak{g}$, and $t \in \mathbb{R}$. First,

$$g\left(\mathrm{e}^{\mathrm{ad}_X}\mathrm{e}^{t\mathrm{ad}_Y}\right)(Y) \in \mathfrak{g}$$

since

$$\left(\mathrm{e}^{\mathrm{ad}_X}\mathrm{e}^{t\mathrm{ad}_Y} - I\right)^l (Y) \in \mathfrak{g}$$

for all $l \in \mathbb{N}$. Therefore whenever the Baker-Campbell-Hausdorff formula holds for X and Y,

$$\log(\mathrm{e}^X \mathrm{e}^Y) \in \mathfrak{g}.$$

By induction,

$$p\left((\mathrm{ad}_X)^l(Y)\right) = (\mathrm{ad}_{p(X)})^l(p(Y))$$

holds for all $l \in \mathbb{N}$, where

$$p(\mathrm{ad}_X(Y)) = p([X,Y]) = [p(X), p(Y)] = \mathrm{ad}_{p(X)}(p(Y))$$

illustrates the base step. Thus

$$\begin{aligned}
p(\mathrm{e}^{t\,\mathrm{ad}_X}(Y)) &= p\left(Y + t[X,Y] + \frac{t^2}{2}[X,[X,Y]] + \frac{t^3}{6}[X,[X,[X,Y]]] + ...\right) \\
&= p(Y) + t[p(X),p(Y)] + \frac{t^2}{2}[p(X),[p(X),p(Y)]] + \frac{t^3}{6}[p(X),[p(X),[p(X),p(Y)]]] + ... \\
&= \mathrm{e}^{t\,\mathrm{ad}_{p(X)}}(p(Y)).
\end{aligned}$$

Similarly,

$$p\left((\mathrm{e}^{\mathrm{ad}_X}\,\mathrm{e}^{t\,\mathrm{ad}_Y} - I)^l(Y)\right) = (\mathrm{e}^{\mathrm{ad}_{p(X)}}\,\mathrm{e}^{t\,\mathrm{ad}_{p(Y)}} - I)^l(p(Y))$$

for all $l \in \mathbb{N}$. Therefore

$$\begin{aligned}
p\left(g\left(\mathrm{e}^{\mathrm{ad}_X}\,\mathrm{e}^{t\,\mathrm{ad}_Y}\right)(Y)\right) &= \sum_{l=0}^{\infty} a_k p\left((\mathrm{e}^{\mathrm{ad}_X}\,\mathrm{e}^{t\,\mathrm{ad}_Y} - I)^l(Y)\right) \\
&= \sum_{l=0}^{\infty} a_k (\mathrm{e}^{\mathrm{ad}_{p(X)}}\,\mathrm{e}^{t\,\mathrm{ad}_{p(Y)}} - I)^l(p(Y)) \\
&= g\left(\mathrm{e}^{\mathrm{ad}_{p(X)}}\,\mathrm{e}^{t\,\mathrm{ad}_{p(Y)}}\right)(p(Y)).
\end{aligned}$$

With this in mind, if the Baker-Campbell-Hausdorff formula holds for X and Y and for $p(X)$ and $p(Y)$, then

$$\begin{aligned}
p\left(\log(\mathrm{e}^X\,\mathrm{e}^Y)\right) &= p(X) + \int_0^1 p\left(g\left(\mathrm{e}^{\mathrm{ad}_X}\,\mathrm{e}^{t\,\mathrm{ad}_Y}\right)(Y)\right) dt \\
&= p(X) + \int_0^1 g\left(\mathrm{e}^{\mathrm{ad}_{p(X)}}\,\mathrm{e}^{t\,\mathrm{ad}_{p(Y)}}\right)(p(Y)) dt \\
&= \log(\mathrm{e}^{p(X)}\,\mathrm{e}^{p(Y)}).
\end{aligned}$$

□

This corollary illustrates why the Baker-Campbell-Hausdorff Formula will come into use. With that said, the outline for Theorem 4.4.9 goes as follows.

(1) Find $N_I \subseteq G$, an open neighborhood about I, and $N_0 \subseteq \mathfrak{g}$, an open neighborhood about 0, such that the matrix exponential maps N_0 homeomorphically onto N_I. Note that $\log|_{N_I} : N_I \to N_0$ provides a local inverse. Furthermore, make sure that N_0 and N_I are small enough so that the Baker-Campbell-Hausdorff formula applies to $\log A$, for all $A \in N_I$.

(2) Starting with the Lie algebra representation $p : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$ for $\mathfrak{g}$, define ρ on the neighborhood N_I by setting

$$\rho(A) := \mathrm{e}^{p(\log A)}.$$

(3) For any $A \in G$, define $\rho(A)$ along a path connecting I to A. Let $\gamma : [0,1] \to G$ such that $\gamma(0) = I$ and $\gamma(1) = A$. Then, by compactness of $[0,1]$, there is an interval partition of $[0,1]$

$$0 = t_0 < t_1 < t_2 ... < t_{k-1} < t_k = 1,$$

such that, for all $l \in [k]$, and $t_{l-1} \leq s \leq t \leq t_l$, one has

$$A_t(A_s)^{-1} = \gamma(t)(\gamma(s))^{-1} \in N_I,$$

where the notation 'A_t' will replace '$\gamma(t)$' for simplcity. Like in Corollary 4.4.6, write

$$A = (A_k A_{k-1}^{-1})(A_{k-1}A_{k-2}^{-1})....(A_2 A_1^{-1})(A_1 A_0),$$

where $A_k = A$ and $A_0 = I$. Using this, set

$$\rho(A) := \rho(A_k A_{k-1}^{-1})\rho(A_{k-1}A_{k-2}^{-1})....\rho(A_2 A_1^{-1})\rho(A_1 A_0),$$

where, by step two, $\rho(A_l A_{l-1}^{-1}) = \mathrm{e}^{p(\log(A_l A_{l-1}^{-1}))}$, for each $l \in [k]$.

(4) Show the assignment $\rho(A)$ is independent of partition. So start by showing that $\rho(A)$ remains unchanged when one refines the partition

$$0 = t_0 < t_1 < t_2 ... < t_{k-1} < t_k = 1.$$

To do this suppose that a point s is added to the original partition between t_{l-1} and t_l, for some l. Then, $A_l A_s^{-1}$ and $A_s A_{l-1}^{-1}$ are in N_I as well. Thus $\rho(A_l A_s^{-1})\rho(A_s A_{l-1}^{-1})$ replaces the odd term $\rho(A_l A_{l-1}^{-1})$. Now set $B = A_l A_s^{-1}$ and $C = A_s A_{l-1}^{-1}$. Then by Corollary 4.4.12,

$$\begin{aligned}
\log\left(\rho(A_l A_s^{-1})\rho(A_s A_{l-1}^{-1})\right) &= \log\left(\mathrm{e}^{p(\log B)}\,\mathrm{e}^{p(\log C)}\right) \\
&= p\left(\log\left(\mathrm{e}^{\log B}\,\mathrm{e}^{\log C}\right)\right) \\
&= p(\log BC) \\
&= p(\log(A_l A_{l-1}^{-1})).
\end{aligned}$$

Indeed, N_I was chosen such that the Baker-Campbell-Hausdorff formula would apply to all $A \in N_I$. Finally,

$$\begin{aligned}
\rho(A_l A_s^{-1})\rho(A_s A_{l-1}^{-1}) &= \mathrm{e}^{\log\left(\rho(A_l A_s^{-1})\rho(A_s A_{l-1}^{-1})\right)} \\
&= \mathrm{e}^{p(\log(A_l A_{l-1}^{-1}))}.
\end{aligned}$$

Thus,

$$\mathrm{e}^{p(\log(A_l A_s^{-1}))}\,\mathrm{e}^{p(\log(A_s A_{l-1}^{-1}))} = \mathrm{e}^{p(\log(A_l A_{l-1}^{-1}))}.$$

Therefore, $\rho(A)$ remains unchanged when one refines the partition. In addition to this, any two partitions have a common refinement. Consequently, $\rho(A)$ is independent of the partition chosen.

(5) Show the assignment $\rho(A)$ is independent of path. G is simply connected, thus any two paths connecting I to A are homotopic. Thus, if γ and Γ denote the two paths, then there exists a continuous map

$$h : [0,1] \times [0,1] \to G,$$

such that, for all $t \in [0,1]$, $h(0,t) = \gamma(t)$ and $h(1,t) = \Gamma(t)$, where $h(s,0) = I$ and $h(s,1) = A$ for all $s \in [0,1]$. Using the compactness of $[0,1] \times [0,1]$, find an integer m such that

$$h(s,t)h(s',t')^{-1} \in N_I,$$

whenever $|t - t'| \leq \frac{2}{m}$ and $|s - s'| \leq \frac{2}{m}$. Now define a sequence of paths

$$\{\zeta_{k,l} : [0,1] \to G \mid 0 \leq k \leq m-1,\ 0 \leq l \leq m\} \cup \{\zeta_{m,0}\}$$

by setting $\zeta_{m,0}(t) = \Gamma(t)$, and for $k < m$

$$\zeta_{k,l}(t) = \begin{cases} h(\frac{k+1}{m}, t) & \text{if } 0 \leq t \leq \frac{l-1}{m} \\ h(\frac{k+l}{m} - t, t) & \text{if } \frac{l-1}{m} \leq t \leq \frac{l}{m} \\ h(\frac{k}{m}, t) & \text{if } \frac{l}{m} \leq t \leq 1 \end{cases}.$$

This sequence of paths should be interpreted as successively deforming γ into Γ step by step. To be exact, starting with $k, l = 0$, one deforms $\zeta_{k,l}$ into $\zeta_{k,l+1}$ for each $0 \leq l \leq m-1$, then one deforms $\zeta_{k,m}$ into $\zeta_{k+1,0}$, and so on until one deforms $\zeta_{m-1,m}$ into $\zeta_{m,0} = \Gamma$. With this, the value of $\rho(A)$ will be shown to be the same regardless of which path, γ or Γ, is used to compute it. This is done by showing that, for each $0 \leq k \leq m-1$, and all $0 \leq l \leq m-1$, the value of $\rho(A)$ computed along $\zeta_{k,l}$ is the same as its value computed along $\zeta_{k,l+1}$, and that the value of $\rho(A)$ computed along $\zeta_{k,m}$ is the same as its value computed along $\zeta_{k+1,0}$, for $0 \leq k \leq m-2$. To see this, notice that, for all $0 \leq k, l < m$, if $t \in [0, \frac{l-1}{m}] \cup [\frac{l+1}{m}, 1]$ then $\zeta_{k,l}(t) = \zeta_{k,l+1}(t)$. So chose a common partition of $[0,1]$ to be

$$0 < \frac{1}{m} < ... < \frac{l-1}{m} < \frac{l+1}{m} < \frac{l+2}{m} < ... < 1,$$

for paths $\zeta_{k,l}$ and $\zeta_{k,l+1}$. Then, $\zeta_{k,l}(t_i)\zeta_{k,l}(t_{i-1})^{-1} = \zeta_{k,l+1}(t_i)\zeta_{k,l+1}(t_{i-1})^{-1} \in N_I$, for all partition point $t_i \in \{\frac{1}{m}, ..., \frac{l-1}{m}, \frac{l+1}{m}, \frac{l+2}{m}, ..., 1\}$, since $|t_i - t_{i-1}| \leq \frac{2}{m}$, and $|s - s'| \leq \frac{1}{m}$ for all applicable s, s'. Thus, the value of $\rho(A)$ is the same for paths $\zeta_{k,l}$ and $\zeta_{k,l+1}$. A similar argument proves this for paths $\zeta_{k,m}$ and $\zeta_{k+1,0}$, for each $0 \leq k \leq m-1$. Therefore, $\rho(A)$ is independent of path.

(6) Show that ρ is a Lie group representation. By the way ρ was defined, its clear that $\rho(AB) = \rho(A)\rho(B)$, and that ρ is smooth.

(7) Show that ρ induces p, i.e. $\dot{\rho} = p$. Let $X \in \mathfrak{g}$. Then, for t sufficiently small, $\mathrm{e}^{tX} \in N_I$. Thus

$$\rho(\mathrm{e}^{tX}) = \mathrm{e}^{p(\log \mathrm{e}^{tX})} = \mathrm{e}^{tp(X)}.$$

Therefore,

$$\begin{aligned}\dot{\rho}(X) &= \frac{d}{dt}\left(\rho(\mathrm{e}^{tX})\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(\mathrm{e}^{tp(X)}\right)\Big|_{t=0}\\ &= p(X),\end{aligned}$$

since $\frac{d}{dt}\left(\rho(\mathrm{e}^{tX})\right)\big|_{t=0} = X$, for any $X \in M_n(\mathbb{C})$.

Again, if $n = \dim V$, then $\mathrm{GL}(V)$ can be identified with $\mathrm{GL}(n,\mathbb{C})$ and $\mathfrak{g}(V)$ can be identified with $\mathfrak{g}(n,\mathbb{C})$. Thus, Theorem 4.4.9 is true for any $p : \mathfrak{g} \to \mathfrak{g}(V))$, whenever G is simply connected.

Corollary 4.4.13. *Let G be a connected and simply connected (complex) matrix Lie group. Then, there is a one to one correspondence between distinct (complex analytic) Lie group representations ρ for G and distinct (complex linear) Lie algebra representations p for $\mathfrak{g}$. This correspondence is given by the property that*

$$\rho(\mathrm{e}^X) = \mathrm{e}^{p(X)}$$

for all $X \in \mathfrak{g}$.

Furthermore, ρ is irreducible if and only if p is irreducible.

Now, the reason for Corollary 4.4.13 is that both $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are simply connected, for all positive integers n. With this in mind, the chapter is finally ready to conclude with the following two main theorems.

Theorem 4.4.14. *Let $\rho : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(V)$ be a complex analytic Lie group representation for $\mathrm{SL}(n,\mathbb{C})$. Then the restriction, $\rho|_{\mathrm{SU}(n)} : \mathrm{SU}(n) \to \mathrm{GL}(V)$ is an irreducible Lie group representation for $\mathrm{SU}(n)$ if and only if $\rho : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(V)$ is irreducible.*

Furthermore, two complex analytic Lie group representation for $\mathrm{SL}(n,\mathbb{C})$, ρ and ϱ, are equivalent if and only if $\rho|_{\mathrm{SU}(n)}$ and $\varrho|_{\mathrm{SU}(n)}$ are equivalent.

Proof. Its clear that if the restriction, $\rho|_{\mathrm{SU}(n)} : \mathrm{SU}(n) \to \mathrm{GL}(V)$ is an irreducible Lie group representation for $\mathrm{SU}(n)$, then ρ is irreducible for $\mathrm{SL}(n,\mathbb{C})$. True! Any $\mathrm{SL}(n,\mathbb{C})-$submodule is also a $\mathrm{SU}(n)-$submodule.

Conversely, suppose that ρ is irreducible for $\mathrm{SL}(n,\mathbb{C})$. Then, by Theorem 4.4.8, the induced Lie Algebra representation $\dot{\rho}$ is irreducible, since $\mathrm{SL}(n,\mathbb{C})$ is connected. Now by Corallary 4.4.4, $\dot{\rho}$ is also complex linear, since ρ is complex analytic. Furthermore, it was established that

$$\mathfrak{su}(n)_{\mathbb{C}} \cong \mathfrak{sl}(n,\mathbb{C}).$$

With this in mind, note that $\dot{\rho}|_{\mathfrak{su}(n)}$ is equal to the induced Lie algebra representation corresponding to the restriction $\rho|_{\mathrm{SU}(n)}$. Thus, $\dot{\rho}$ is the unique complex extension of induced Lie algebra representation corresponding to the restriction $\rho|_{\mathrm{SU}(n)}$. Hence, $\dot{\rho}|_{\mathfrak{su}(n)}$ is also irreducible, by Theorem 4.3.3. Finally, using Theorem 4.4.8 a second time, one sees that the restriction $\rho|_{\mathrm{SU}(n)}$ is an irreducible Lie group representation for the connected matrix Lie group $\mathrm{SU}(n)$.

Now, let ρ and ϱ be two complex analytic Lie group representation for $\mathrm{SL}(n,\mathbb{C})$. If ρ and ϱ are equivalent, then clearly $\rho|_{\mathrm{SU}(n)}$ and $\varrho|_{\mathrm{SU}(n)}$ are equivalent.

So suppose the converse. By Theorem 4.4.8, $\dot{\rho}|_{\mathrm{SU}(n)}$ and $\dot{\varrho}|_{\mathrm{SU}(n)}$ are equivalent. Also, by Theorem 4.3.3, their complex extensions to $\mathfrak{sl}(n,\mathbb{C})$ are equivalent as well. But these clearly co-inside with $\dot{\rho}$ and $\dot{\varrho}$, respectively. Thus, $\dot{\rho}$ and $\dot{\varrho}$ are equivalent for $\mathfrak{sl}(n,\mathbb{C})$. Finally, a second use of Theorem 4.4.8 shows that ρ and ϱ are equivalent as complex analytic Lie group representation for $\mathrm{SL}(n,\mathbb{C})$. □

Theorem 4.4.15. *Let $\rho : \mathrm{SU}(n) \to \mathrm{GL}(V)$ be an irreducible Lie group representation for $\mathrm{SU}(n)$. Then there exists a unique complex analytic Lie group representation $\varrho : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(V)$ such that*

$$\varrho|_{\mathrm{SU}(n)} = \rho.$$

Moreover, let q be the unique complex extension of $\dot{\rho}$ for $\mathfrak{sl}(n,\mathbb{C}) \cong \mathfrak{su}(n)_{\mathbb{C}}$. Then the representation ϱ is determined by the property that

$$\varrho(\mathrm{e}^{X}) = \mathrm{e}^{q(X)}$$

for all $X \in \mathfrak{sl}(n,\mathbb{C})$. In particular, if $A \in \mathrm{SL}(n,\mathbb{C})$, then

$$\varrho(A) = \mathrm{e}^{q(X_1)}\,\mathrm{e}^{q(X_2)} \ldots \mathrm{e}^{q(X_k)}$$

for any collection of $\{X_l\} \subseteq \mathfrak{sl}(n,\mathbb{C})$ such that $A = \mathrm{e}^{X_1}\,\mathrm{e}^{X_2} \ldots \mathrm{e}^{X_k}$.

Proof. Suppose ρ is irreducible. Then $\dot{\rho} : \mathfrak{su}(n) \to \mathfrak{gl}(V)$ irreducible by Theorem 4.4.8, since $\mathrm{SU}(n)$ is connected. Now let $q : \mathfrak{sl}(n,\mathbb{C}) \to \mathfrak{gl}(V)$ be the unique complex extension of $\dot{\rho}$. Then by Theorem 4.3.3, q is also irreducible. Now, $SL(n,\mathbb{C})$ is simply connected. Thus by Corollary 4.4.13, there exists a unique Lie group representation

$$\varrho : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{GL}(V)$$

of $\mathrm{SL}(n,\mathbb{C})$ such that

$$\varrho(\mathrm{e}^{tX}) = \mathrm{e}^{tq(X)}$$

for all $X \in \mathfrak{sl}(n,\mathbb{C})$. Also, it is clear that $\varrho|_{\mathrm{SU}(n)} = \rho$

Finally by Corollary 4.4.4, ϱ is a complex analytic Lie group representation, since $\dot{\varrho} = q$ is a complex linear Lie algebra representation. □

Arriving at Theorem 4.4.14 and Theorem 4.4.15 the overall goal for this section has been fulfilled. That is, it has been established that distinct, irreducible Lie group representations of $\mathrm{SU}(n)$ are in one to one correspondence with distinct, irreducible complex analytic Lie group representations of $\mathrm{SL}(n,\mathbb{C})$.

With this result fully justified, the next chapter specifically investigates complex analytic representations for $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$. To briefly explain, it will shown that these complex analytic representations are uniquely identified by their *highest weights*, which are particular analytic homomorphisms that appear as a consequence of the representations. This is important since these highest weights are in a one to one correspondence with the integer partitions that characterize the irreducible tensor representations defined in Chapter 3.

Chapter 5

Highest weight description of analytic representations

Last chapter, the finite dimensional Lie group representations of $\mathrm{SU}(n)$ were placed in one to one correspondence with the finite dimensional complex analytic representations of $\mathrm{SL}(n,\mathbb{C})$. What remains is linking such representations of $\mathrm{SL}(n,\mathbb{C})$ to integer partitions, and establishing the setting of the irreducible tensor representations of $\mathrm{GL}(n,\mathbb{C})$ as the source for the realizations of the desired irreducibles of $\mathrm{SU}(n)$. Now, what leads to these connections is the unique structure of the complex matrix Lie groups, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$, in combination with the essential features of their complex analytic representations. Produced from the complex analytic nature of such representations of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$, necessary objects called *weights* and *weight spaces* will be shown to be directly determined by n-tuples of integers. Most crucial is that the irreducibles are identifiable by *highest weights*, which are themselves given by some appropriate weakly increasing sequence of integers. Ultimately these relationships will allow $\mathrm{SL}(n,\mathbb{C})$ to provide the final tie between the irreducible tensor representations of $\mathrm{GL}(n,\mathbb{C})$ and the all the finite dimensional irreducible Lie group representations of $\mathrm{SU}(n)$. Therefore the overall aim of this chapter will be to provide an adequate description of such complex analytic representations to the Matrix Lie groups, $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$.

The methods that follow in this chapter have been adapted from a combination of techniques provided by the treatment given by Sternberg [4]. Furthermore, the phrase 'finite dimensional' will be an understood quality of the complex analytic representations, and therefore, will be omitted.

5.1 Lie's theorem

In short, weight spaces are essentially simultaneous eigenspaces relative to the subgroup of diagonal matrices of either $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{C})$ that appear in modules carrying complex analytic representations belonging to the two complex Matrix Lie groups. Lie's Theorem now provides a key observation in defining and finding weights and weight spaces for such representations of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$.

Let G be a group. The *derived series* $(G^{(k)})_k$ is defined by

$$\begin{aligned} G^{(0)} &:= G \\ G^{(k)} &:= [G^{(k-1)}, G^{(k-1)}] \quad k \geq 1. \end{aligned}$$

If $G^{(l)} = \{\epsilon\}$ for some l, then G is *solvable*. The smallest such l is the *solvable length* of G.

Theorem 5.1.1 (Lie's Theorem). *Let G be a connected, solvable Lie group, and suppose $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation for G. If V is finite-dimensional, then* $\dim V = 1$.

Before the proof consider the following.

Definition 5.1.2. Let $\rho : G \to \mathrm{GL}(V)$ be a representation for G. A *simultaneous eigenvalue* and the corresponding *simultaneous eigenspace* for the representation is a pair (μ, V_μ) such that

(1) $\mu : G \to \mathbb{C}^*$ is a continuous homomorphism, and

(2) $V_\mu = \{v \in V \mid \rho(g)v = \mu(g)v \, \forall g \in G\}$ is a non-trivial subspace in V.

Furthermore, a *simultaneous eigenvector* v is a non-zero element of a simultaneous eigenspace.

Lemma 5.1.3. *Let k be a positive integer. Then, for any set of k distinct simultaneous eigenvalues of a representation $\rho : G \to \mathrm{GL}(V)$, the corresponding simultaneous eigenspaces are linearly independent. In particular, the number of distinct simultaneous eigenvalues for a representation cannot exceed the dimension of V.*

Proof. This will be shown through induction on the size of distinct simultaneous eigenvalues. The lemma is clearly true for $k = 1$. So suppose, for some $l \geq 1$, the lemma is true for any set of l distinct simultaneous eigenvalues of ρ, and let $\{\mu_i\}_{i \in [l+1]}$ be a set of distinct simultaneous eigenvalues with their corresponding simultaneous eigenspaces being $\{V_i\}_{i \in [l+1]}$. For each $i \in [l+1]$, choose any element $v_i \in V_i \setminus \{0_V\}$, and suppose that, for some set of complex numbers $\{a_i\}_{i \in [l+1]}$,

$$a_1 v_1 + a_2 v_2 + ... + a_{l+1} v_{l+1} = 0_V.$$

Now, consider the map

$$\chi : G \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

defined by

$$\chi(g) = \rho(g) - \mu_{l+1}(g).$$

Realize that, for all $i \in [l+1]$, and $g \in G$, it is true that $\rho(g)v_i = \mu_i(g)v_i$. Thus $\chi(g)v_{l+1} = 0_V$, and

$$\chi(g)(a_1 v_1 + a_2 v_2 + ... + a_{l+1} v_{l+1}) = \sum_{i=1}^{l} a_i(\mu_i(g) - \mu_{l+1}(g))v_i = 0_V.$$

By the linear independence of the first l simultaneous eigenvectors (the inductive hypothesis), one has

$$a_i(\mu_i(g) - \mu_{l+1}(g)) = 0$$

for all $i \in [l]$ and $g \in G$.

On the other hand, it was assumed that all the simultaneous eigenvalues are distinct. Consequently, for each $i < l+1$, there is some $g_i \in G$ such that

$$\mu_i(g_i) \neq \mu_{l+1}(g_i).$$

Thus,

$$a_1 = a_2 = ... = a_l = 0.$$

Finally considering above, $a_{l+1} = 0_V$ as well. Therefore $\{V_i\}_{i \in [l+1]}$ are linearly independent as claimed. □

With the previous lemma given, here is the proof of Lie's Theorem.

Proof. The method will be induction on the solvable length of G. Throughout the proof G is assumed to be a connected Lie group. If the solvable length of G is 1, then G is abelian. Suppose $\rho : G \to \mathrm{GL}(V)$ is a finite irreducible representation for G. Let $g \in G$, and note that $\rho(g) \in \mathrm{Hom}_G(V, V)$ since G is abelian. By Schur's Lemma, $\rho(g) = \lambda_g \cdot \mathrm{id}_V$ for some $\lambda_g \in \mathbb{C}$. Thus, any one dimensional subspace is invariant, and is therefore equal to V.

Suppose the theorem is true for all connected Lie groups with a solvable length $k \geq 1$, and suppose that G has a solvable length $k+1$. Then the solvable length of $G^{(1)}$ is k. Furthermore, $G^{(1)}$ is connected. Indeed, let $h \in G^{(1)}$. Then, for some $m \in \mathbb{N}$ and collection $\{g_i\}_{i \in [m]} \subseteq G$,

$$h = g_1 g_2 ... g_m \, g_1^{-1} g_2^{-1} ... g_m^{-1}.$$

But G is connected. Thus for each $i \in [m]$, there is a continuous function

$$\gamma_i : [0,1] \to G$$

such that $\gamma_i(0) = \epsilon$ and $\gamma_i(1) = g_i$. As a result, the function $\gamma : [0,1] \to G^1$ defined by

$$t \mapsto \gamma_1(t)\gamma_2(t)...\gamma_m(t)\gamma_1(t)^{-1}\gamma_2(t)^{-1}...\gamma_m(t)^{-1}$$

is a continuous path connecting ϵ to h.

Now $\rho \mid_{G^{(1)}}: G^{(1)} \to \mathrm{GL}(V)$, in general, won't be irreducible. However, V will contain some invariant subspace for $G^{(1)}$, which by the inductive hypothesis, will be one dimensional. Thus there is some simultaneous eigenvalue μ and eigenspace V_μ for $\rho \mid_{G^{(1)}}$.

Let $g \in G$, $h \in G^{(1)}$, and $v_\mu \in V_\mu$. Recall that $G^{(1)}$ is normal. Thus

$$\begin{aligned} \rho(h)(\rho(g)v_\mu) &= \rho(g)(\rho(g^{-1})\rho(h)\rho(g)v_\mu) \\ &= \rho(g)(\rho(g^{-1}hg)v_\mu) \\ &= \mu(g^{-1}hg)\rho(g)v_\mu. \end{aligned}$$

Define $\mu_g : G^1 \to \mathbb{C}^*$ by

$$\mu_g(h) = \mu(g^{-1}hg).$$

Because conjugation by g is an automorphism of $G^{(1)}$, μ_g is another weight. As a result,

(μ_g, V_{μ_g}) is a simultaneous eigenvalue/eigenspace pair for $\rho\mid_{G^{(1)}}$ with the property that

$$V_{\mu_g} = \rho(g).V_\mu.$$

Still considering the original μ, apply Lemma 5.1.3 to the collection

$$\Omega := \{\mu_g : G^{(1)} \to \mathbb{C}^* \mid g \in G\},$$

to conclude $|\Omega| \leq \dim V$. In particular, Ω must be finite. Fix an $h \in G^{(1)}$, and consider the map $\varphi_h : G \to \mathbb{C}^*$ given by $\varphi_h(g) = \mu_g(h)$. Note that $\varphi_h(G)$ is discrete since Ω is finite. Importantly, the assumed continuity of the representation ρ implies the continuity of μ. Therefore the map φ_h is continuous, and by the connectedness of G, $\varphi_h(G) = \{z\}$ for some $z \in \mathbb{C}^*$. In fact, more is true! Note that $\varphi_h(e) = \mu(h)$. Thus $z = \mu(h)$, and consequently,

$$\mu_g(h) = \mu(h)$$

for all $g \in G$ and all $h \in G^{(1)}$. Thus for any $g \in G$ and $h \in G^1$,

$$\rho(h)(\rho(g)v_\mu) = \mu(h)\rho(g)v_\mu$$

for all $v_\mu \in V_\mu$. From this, $V_\mu = V$, and since V_μ is invariant, $V_\mu = V$. Therefore

$$\rho(h) = \mu(h) \cdot \mathrm{id}_V$$

for each $h \in G^{(1)}$.

Now, note two facts:

(1) For all $g_1, g_2 \in G$, there is some $h \in G^{(1)}$ such that $hg_2g_1 = g_1g_2$, and

(2) The isomorphism $\rho(g_1)$ has an eigenvalue(nonzero)/eigenvector pair (λ, w).

Point (1) is easily seen by $[g_1, g_2] \in G^1$, and point (2) follows since $\mathbb{C}$ is algebraically closed. One now has

$$\begin{aligned} \rho(g_1)(\rho(g_2)w) &= \rho(h)\rho(g_2)\rho(g_1)w \\ &= \lambda\mu(h)(\rho(g_2)w). \end{aligned}$$

Thus $\rho(g_2)w$ is also an eigenvector of $\rho(g_1)$ with eigenvalue $\lambda\mu(h)$, and $h = [g_1, g_2]$. Let $\zeta : G \to \mathbb{C}^*$ be defined by

$$\zeta(g) = \lambda\mu([g_1, g_2]).$$

Like the map φ_h described above, ζ is continuous. However, $\rho(g_1)$ can have only finitely many distinct eigenvalues. Therefore since G is connected, $\zeta(G) = \{z\}$ for some complex z.

Finally, $\zeta(g_1) = \lambda\mu([g_1, g_1]) = \lambda$. Thus for all $g \in G$, $\zeta(g) = \lambda$. Consequently

$$\begin{aligned} \rho(g_1)(\rho(g)w) &= \lambda\rho(g)w \\ &= \rho(g)(\rho(g_1)w) \\ &= \lambda\rho(g)w \end{aligned}$$

for all $g \in G$. By Schur's Lemma,

$$\rho(g_1) = \lambda \cdot \mathrm{id}_V .$$

Hence, for every $g \in G$, there would be some $\lambda_g \in \mathbb{C}^*$ such that

$$\rho(g) = \lambda_g \cdot \mathrm{id}_V .$$

Therefore $\dim V = 1$ since any one dimensional subspace would be invariant under G. $\square$

5.2 Gauss decomposition of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{C})$

Before the application Lie's Theorem to complex analytic representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, this section provides a description of the unique structure of the two matrix Lie groups. Both groups contain dense (relatively) open subsets of matrices that are decomposable into products of triangular unipotent matrices with diagonal matrices. A matrix is *upper (lower) triangular unipotent* if it has all ones down the diagonal and is upper (lower) triangular. Lie's Theorem, combined with this factorization, will facilitate the *highest weight* identification of irreducible complex analytic representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$. In this process, four other matrix Lie groups will be participating:

$$\begin{aligned}
\mathcal{Z} &= \{D \in \mathrm{GL}(n, \mathbb{C}) \mid D = \mathrm{diag}(\delta_1, \delta_2, ..., \delta_n)\} \\
\mathcal{D} &= \mathcal{Z} \cap \mathrm{SL}(n, \mathbb{C}) \\
\mathcal{U} &= \{U \in \mathrm{GL}(n, \mathbb{C}) \mid U \text{ is upper triangular unipotent}\} \\
\mathcal{L} &= \{L \in \mathrm{GL}(n, \mathbb{C}) \mid L \text{ is lower triangular unipotent}\}.
\end{aligned}$$

Now, it is straightforward to verify that $\mathcal{U}$ and $\mathcal{L}$ are subgroups of $\mathrm{SL}(n, \mathbb{C})$. Furthermore, by appealing to the sequential characterization of closed sets, one can quickly establish that these four are additionally closed subsets of $\mathrm{GL}(n, \mathbb{C})$, making them matrix Lie groups.

Proposition 5.2.1. *All the matrix Lie groups mentioned thus far are (path) connected.*

Proof. First, $\mathcal{Z}$ is homeomorphic to $(\mathbb{C}^*)^n$. Thus $\mathcal{Z}$ is (path) connected. In the same manner, $\mathcal{D}$ is homeomorphic to $(\mathbb{C}^*)^{n-1}$. Thus the result is true for $\mathcal{D}$. Furthermore, as topological spaces,

$$\mathcal{U} \cong \mathcal{L} \cong \mathbb{C}^{\frac{n(n-1)}{2}}.$$

This establishes the claim for these two as well.

Let $A \in \mathrm{GL}(n, \mathbb{C})$. Then A is similar to some upper triangular matrix

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}.$$

since the characteristic polynomial of A is in $\mathbb{C}[t]$, and thus factors completely. Note that the set of upper triangular complex matrices is homeomorphic to the product $(\mathbb{C}^*)^n \times \mathbb{C}^{\frac{n(n-1)}{2}}$. Thus one can find a continuous path $\gamma : [0,1] \to \mathrm{GL}(n,\mathbb{C})$ that connects I to T. Now, $A = P^{-1}TP$ for some fixed $P \in \mathrm{GL}(n,\mathbb{C})$. Consequently

$$t \to P^{-1}\gamma(t)P$$

defines a path connecting I to A. Therefore $\mathrm{GL}(n,\mathbb{C})$ is path connected since every matrix is connected, by some path, to the identity matrix.

Proving that $\mathrm{SL}(n,\mathbb{C})$ is path connected follows analogously to the case for $\mathrm{GL}(n,\mathbb{C})$. However, one references

$$(\mathbb{C}^*)^{n-1} \times \mathbb{C}^{\frac{n(n-1)}{2}}$$

instead of $(\mathbb{C}^*)^n \times \mathbb{C}^{\frac{n(n-1)}{2}}$. □

Here is the formal introduction of the needed factorization.

Definition 5.2.2. A *Gauss decomposition* of a matrix $A \in \mathrm{GL}(n,\mathbb{C})$ is a factorization of A of either of the following forms

$$A = LDU \quad \text{or} \quad A = UDL$$

for some $L \in \mathcal{L}$, $D \in \mathcal{Z}$, and $U \in \mathcal{U}$.

Proposition 5.2.3. *Let $A \in \mathrm{GL}(n,\mathbb{C})$. If A has a Gauss decomposition of either form, then the decomposition is unique.*

Proof. Let $L_i \in \mathcal{L}$, $D_i \in \mathcal{Z}$, and $U_i \in \mathcal{U}$, for $i = 1,2$. Suppose that $A \in \mathrm{GL}(n,\mathbb{C})$ can be factored into $A = L_iD_iU_i$ for each $i = 1,2$. Then

$$L_1D_1U_1 \;=\; L_2D_2U_2.$$

Thus

$$D_2^{-1}L_2^{-1}L_1D_1 \;=\; U_2U_1^{-1}.$$

However, $U_2U_1^{-1} = I$ since it would be both upper and lower triangular unipotent. Hence $U_1 = U_2$, and as a result,

$$L_1 = L_2 \quad \text{and} \quad D_1 = D_2.$$

The uniqueness of the factorization of A into UDL follows analogously. □

Necessary and sufficient conditions for a Gauss decomposition of a matrix will be presented in the following lemma. First, it will be convenient to define, for each $k \in [n]$,

$$\Delta_k(A) = \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

along with

$$\hat{\Delta}_k(A) = \det \begin{bmatrix} a_{(n-k+1)(n-k+1)} & \cdots & a_{(n-k+1)(n-1)} & a_{(n-k+1)n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)(n-k+1)} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n(n-k+1)} & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

For reference, call $\Delta_k(A)$ the kth *leading principal minor* of A, and $\hat{\Delta}_k(A)$ the kth *trailing principal minor of* A. Also, observe that UD and DU are both upper triangular with the same diagonal as D. Likewise, LD and DL are both lower triangular with the same diagonal as D.

Lemma 5.2.4. *Let $A \in \mathrm{GL}(n, \mathbb{C})$. Then A has a Gauss decomposition of the form LDU if and only if all the leading principal minors of A are non-vanishing.*

Furthermore, A has a Gauss decomposition of the form UDL if and only if all the trailing principal minors of A are non-vanishing.

Proof. Suppose that $A = LDU$, and let $C = (LD)^{-1}$ be given by

$$C = \begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

Now, $CA = U$. So, by computing the matrix multiplication and observing the equality assumed per entry of

$$\begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

one generates n systems of linear equations. Notice that the kth linear equation is

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kk} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

(For clarification, there are all zeros above the 1 in the column vector on the right hand side of the equation.)

Each $\Delta_k(A)$ was constructed to compute the kth leading principal minor of A. Furthermore, uniqueness of such a decomposition for A was just verified in Proposition 5.2.3. Consequently, each $\Delta_k(A) \neq 0$. Otherwise, there would be more than one solution, contradicting the uniqueness property.

Conversely, if all the principal minors are non-vanishing, i.e. $\Delta_k(A) \neq 0$, for all $k \in [n]$, then (not assuming the decomposition exists) each of the n systems

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} c_{k1} \\ c_{k2} \\ \vdots \\ c_{kk} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

will have a unique solution. Thus allowing one to first construct the lower triangular matrix C. Afterward, the creation of U, the upper triangular unipotent matrix, would follow, and would be equal to the product CA.

Lastly, let D be diagonal matrix formed by using the diagonal of C^{-1}. Then $L = C^{-1}D^{-1}$ would be lower triangular unipotent. Thus $A = LDU$. Therefore A would have a Gauss decomposition.

To finish the proof, one needs to establish equivalence of the factorization of A into UDL with the property that all the trailing principal minors are non-vanishing. To do this, one simply repeats the first part of the proof with each of the $\hat{\Delta}_k(A)$s instead of the $\Delta_k(A)$s, and consults a new collection of n systems of linear equations using a matrix $B = (UD)^{-1}$. $\square$

The results from Lemma 5.2.4 will soon be expanded on in Theorems 5.2.7 and 5.2.8. Before doing so however, the following additional declaration of matrix groups will be needed:

$$\begin{aligned} \mathcal{A}_+ &:= \mathcal{Z}\mathcal{U} && (5.2.1) \\ \mathcal{A}_- &:= \mathcal{L}\mathcal{Z} && (5.2.2) \\ \mathcal{B}_+ &:= \mathcal{D}\mathcal{U} && (5.2.3) \\ \mathcal{B}_- &:= \mathcal{L}\mathcal{D} && (5.2.4) \end{aligned}$$

Note that these are connected matrix Lie groups. Indeed, one can easily verify that $\mathcal{A}_+$ and $\mathcal{B}_+$ are subgroups of upper triangular matrices in $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$, respectively. Similarly, $\mathcal{A}_-$ and $\mathcal{B}_-$ are easily seen to be the corresponding subgroups of lower triangular matrices in $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$, respectively. Each is closed topologically, using the sequential characterization of closed sets. Path connectedness results considering Proposition 5.2.1.

Finally, since both the set of upper triangular matrices and the set of lower triangular matrices are subgroups in the respective settings of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$,

$$\begin{aligned} \mathcal{Z}\mathcal{U} &= \mathcal{U}\mathcal{Z} \\ \mathcal{L}\mathcal{Z} &= \mathcal{Z}\mathcal{L} \\ \mathcal{D}\mathcal{U} &= \mathcal{U}\mathcal{D} \\ \mathcal{L}\mathcal{D} &= \mathcal{D}\mathcal{L}. \end{aligned}$$

Indeed, from group theory, one has the property that the set product of two subgroups is again a subgroup if and only if the corresponding set product commutes.

Now, the importance of the matrix Lie groups $\mathcal{Z}$, $\mathcal{D}$ and the groups listed in lines 5.2.1

through 5.2.4 is that Lie's Theorem applies to each. Ultimately, from this application, comes the definitions of *weights* and the corresponding *weight spaces*. Therefore the next lemma verifies that the matrix Lie groups previously stated are all solvable.

Lemma 5.2.5. *The commutator subgroup of both $\mathcal{A}_+$ and $\mathcal{B}_+$ is equal to $\mathcal{U}$. Likewise, the commutator subgroup of both $\mathcal{A}_-$and $\mathcal{B}_-$ is equal to $\mathcal{L}$.*

Proof. For this proof, one needs to show that

$$[\mathcal{A}_+,\mathcal{A}_+] = [\mathcal{B}_+,\mathcal{B}_+] = \mathcal{U} \quad \text{and} \quad [\mathcal{A}_-,\mathcal{A}_-] = [\mathcal{B}_-,\mathcal{B}_-] = \mathcal{L}.$$

Now it suffices to just demonstrate that $[\mathcal{A}_+,\mathcal{A}_+] = \mathcal{U}$ since the result does not depend on on the value of the determinant. Additionally, the case for $\mathcal{A}_-$ and $\mathcal{L}$ is completely analogous to the case for $\mathcal{A}_+$ and $\mathcal{U}$ in that the former is just the 'transposed' setting of the latter.

Let A and T be in $\mathcal{A}_+$, with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}.$$

It is easy to verify that each of the diagonal entries of A^{-1} and T^{-1} are just the multiplicative inverses of the corresponding diagonal entry in A and T, respectively. Furthermore, due to being upper triangular, for some collection of constants $\{r_{ij}\}$ and $\{s_{ij}\}$,

$$AT = \begin{bmatrix} a_{11}t_{11} & s_{12} & \cdots & s_{1n} \\ 0 & a_{22}t_{22} & \cdots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}t_{nn} \end{bmatrix} \quad \text{and} \quad A^{-1}T^{-1} = \begin{bmatrix} a_{11}^{-1}t_{11}^{-1} & r_{12} & \cdots & r_{1n} \\ 0 & a_{22}^{-1}t_{22}^{-1} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{-1}t_{nn}^{-1} \end{bmatrix}$$

To be exact, the diagonal entries of the product of two upper triangular matrices are just the product of their respective diagonal entries. Thus, for some collection of constants $\{u_{ij}\}$,

$$ATA^{-1}T^{-1} = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Therefore

$$[\mathcal{A}_+,\mathcal{A}_+] \leq \mathcal{U}$$

since $\mathcal{U}$ contains all the generators of $[\mathcal{A}_+,\mathcal{A}_+]$.

To see the reverse inclusion

$$[\mathcal{A}_+,\mathcal{A}_+] \geq \mathcal{U},$$

appeal to generators as well. Now, $\mathcal{U}$ has generators consisting of *upper transvections*

$$\{T_{ij}(a) \mid a \in \mathbb{C},\, 1 \leq i < j \leq n\}.$$

The transvection $T_{ij}(a)$ is commonly defined to be

$$T_{ij}(a) = I + aE_{ij}.$$

(Recall that E_{ij} is the matrix with a 1 in the (i,j)th entry and zeros elsewhere.) In other words, $T_{ij}(a)$ has all 1s down the diagonal, the constant a in the (i,j)th entry, and zeros elsewhere. The following formula for the various commutators of the $T_{ij}(a)$s is given by

$$[T_{ij}(a), T_{lk}(b)] = I + ab(\delta_{lj}E_{ik} - \delta_{ik}E_{lj}),$$

where $i < j$ and $l < k$ with $a, b \in \mathbb{C}$. Consequently since the upper transvections are also in $\mathcal{A}_+$, for each $1 \leq i < i+1 < j \leq n$, and $a \in \mathbb{C}$,

$$T_{ij}(a) = [T_{ii+1}(a), T_{i+1j}(1)] \in [\mathcal{A}_+, \mathcal{A}_+].$$

For the case when $2 \leq l+1 \leq n$, let $D \in \mathcal{A}_+$ be the diagonal matrix with i in the lth diagonal entry, a $-i$ in the $l+1$th diagonal entry, and with 1s elsewhere along the diagonal. Then

$$[T_{ll+1}\left(\frac{a}{2}\right), D] = T_{ll+1}(a).$$

As a result, for each $l \in [n-1]$, and $a \in \mathbb{C}$,

$$T_{ll+1}(a) \in [\mathcal{A}_+, \mathcal{A}_+].$$

Therefore

$$[\mathcal{A}_+, \mathcal{A}_+] \geq \mathcal{U},$$

since $[\mathcal{A}_+, \mathcal{A}_+]$ contains all the generators of $\mathcal{U}$.

All the other cases follow analogously to this one. Also, note that D, the diagonal matrix introduced in later part of the proof had a determinant equal to one, making it acceptable in the case of $\mathcal{B}_+$ and $\mathcal{B}_-$. □

Proposition 5.2.6. *The matrix Lie groups $\mathcal{U}$ and $\mathcal{L}$ are solvable. In particular, $\mathcal{A}_+, \mathcal{A}_-, \mathcal{B}_+$, and $\mathcal{B}_-$ are all solvable.*

Proof. By Lemma 5.2.5, if $\mathcal{U}$ and $\mathcal{L}$ are solvable, then $\mathcal{A}_+$, $\mathcal{A}_-$, $\mathcal{B}_+$, and $\mathcal{B}_-$ are all solvable as well. Indeed, their solvable lengths would be one unit more than the solvable lengths of $\mathcal{U}$ and $\mathcal{L}$, respectively.

So, for each $k \in [n-1]$, define

$$\mathcal{N}_k \leq \mathcal{U}$$

to be the set of matrices with all zeros in the k diagonals above their main diagonal. For reference, note that $\mathcal{N}_{n-1} = \{I\}$. Let U and C be in $\mathcal{U}$. Then each entry of the diagonal just above the main diagonal of their product UC is equal to the sum of the corresponding entries of U and C. Furthermore, each entry of the diagonal just above the main diagonal of their inverses U^{-1} and C^{-1} is the multiplicative inverse of the corresponding entry in U and C, respectively. With this observation,

$$UCU^{-1}C^{-1} \in \mathcal{N}_1.$$

Thus

$$[\mathcal{U},\mathcal{U}] \leq \mathcal{N}_1.$$

From here, induction, with a similar argument used just previously, shows that

$$UCU^{-1}C^{-1} \in \mathcal{N}_{k+1}$$

whenever $U, C \in \mathcal{N}_k$. Consequently $\mathcal{U}^{(k)}$, the kth derived subgroup, is contained in $\mathcal{N}_k$. Therefore $\mathcal{U}$ is solvable with a solvable length no larger than $n-1$.

Finally, $\mathcal{L}$ is solvable, shown by repeating the previous argument with

$$\mathcal{N}_k \leq \mathcal{L},$$

defined now to be all the matrices with only zeros in the k diagonals below their main diagonal. □

This section concludes with establishing the topological claims concerning the set of matrices having Gauss decompositions. The true utility of these properties will be in verifying the invaluable result presented in Lemma 5.3.5, and Theorem 5.3.8 in Section 5.3.

Theorem 5.2.7. *Both* $\mathcal{LZU}$ *and* $\mathcal{UZL}$ *are dense in* $\mathrm{GL}(n,\mathbb{C})$.

Proof. For each Δ_k, set $U_k = \{A \in M_n(\mathbb{C}) \mid \Delta_k(A) \neq 0\}$. Then, as the compliment of the zero set of a polynomial operation

$$\Delta_k : M_n(\mathbb{C}) \to \mathbb{C},$$

U_k is an open dense subset in $M_n(\mathbb{C})$. Indeed, the corresponding zero set is closed and nowhere dense in $M_n(\mathbb{C})$. Therefore, since the finite intersection of open dense sets is still open and dense,

$$\mathcal{LZU} = \bigcap_{k=1}^{n} U_k$$

is open and dense in $M_n(\mathbb{C})$.

If $A \in \mathrm{GL}(n,\mathbb{C})$, and V is an open neighborhood of A in $\mathrm{GL}(n,\mathbb{C})$, then V is also open in $M_n(\mathbb{C})$. True, $\mathrm{GL}(n,\mathbb{C}) = U_n$ is open as well. By this, V has a non empty intersection with $\bigcap_{k=1}^{n} U_k$. Thus V contains an element of the form $LDU \in \mathcal{LZU}$. Therefore $\mathcal{LZU}$ is dense in $\mathrm{GL}(n,\mathbb{C})$.

Finally, the density of $\mathcal{UZL}$ in $\mathrm{GL}(n,\mathbb{C})$ follows in a similar fashion by repeating the previous argument with the $\hat{\Delta}_k$s. □

Theorem 5.2.8. *Both* $\mathcal{LDU}$ *and* $\mathcal{UDL}$ *are dense in* $\mathrm{SL}(n,\mathbb{C})$.

Proof. This case is a little more problematic since $\mathrm{SL}(n,\mathbb{C})$ is not open in $M_n(\mathbb{C})$ like $\mathrm{GL}(n,\mathbb{C})$. One will need to appeal to $\mathrm{SL}(n,\mathbb{C})$ as complex manifold.

Consider the operations Δ_k and $\hat{\Delta}_k$ for $k \in [n-1]$. By Lemma 5.2.4, a matrix $A \in \mathrm{SL}(n,\mathbb{C})$ is in $\mathcal{LDU}$ if and only if $\Delta_k(A) \neq 0$ for each $k \in [n-1]$. By taking the product

$$P := \prod_{k=1}^{n-1} \Delta_k,$$

one has $A \notin \mathcal{LDU}$ if and only if $P(A) = 0$. Furthermore, P is polynomial in the entries of A. Thus P a global complex analytic function on $\mathrm{SL}(n,\mathbb{C})$.

With that said, suppose that $\mathcal{LDU}$ is not dense in $\mathrm{SL}(n,\mathbb{C})$. Then, for some $A \in \mathrm{SL}(n,\mathbb{C})$, there exists an open neighborhood $N' \subseteq \mathrm{SL}(n,\mathbb{C})$ about A such that $N' \cap \mathcal{LDU} = \emptyset$. Consequently P vanishes completely on the open set N'.

Now consider the atlas for $\mathrm{SL}(n,\mathbb{C})$ introduced in Corollary 4.4.3

$$\{(N_A, \varphi_A) \mid A \in \mathrm{SL}(n,\mathbb{C})\}.$$

Since $\mathrm{SL}(n,\mathbb{C})$ is path connected, one can use the compactness of the interval $[0,1]$ to find a finite collection $\{A_i \mid i \in [K]\}$, such that $A_0 = I$, $A_K = A$ and $N_{A_{i-1}} \cap N_{A_i} \neq \emptyset$ for each $i \in [K]$. With this mind, P vanishes on the open neighborhood N_A since it vanishes on the open set $N' \cap N_A \subseteq N'$. Indeed, a complex analytic function that vanishes on an open subset of its domain must also vanish completely on the connected component containing that open subset. So applying this fact to the pullback

$$P \circ \varphi_A^{-1} : \varphi_A(N_A) \to \mathbb{C},$$

one finds that $P \circ \varphi_A^{-1}$ vanishes on connected open set $\varphi_A(N_A)$ since it vanishes on the open subset $\varphi_A(N_A \cap N')$. Therefore P must be identically zero on N_A.

Continuing, apply this argument to the open subset $N_{A_{K-1}} \cap N_A$ to see that P vanishes on $N_{A_{K-1}}$ as well. So inductively, one establishes that P is zero on all of $N_{A_0} = N_I$. However, this is impossible since $P(I) = 1$. With this contradiction, $\mathcal{LDU}$ is dense in $\mathrm{SL}(n,\mathbb{C})$.

Finally, by repeating the previous argument with the use of the collection $\hat{\Delta}_k$, $k \in [n-1]$, one establishes that $\mathcal{UDL}$ is dense in $\mathrm{SL}(n,\mathbb{C})$ as well. □

5.3 Highest weight identification of irreducible representations

Consider a complex analytic representation of $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{C})$. By the results of Section 5.2, one could potentially apply Lie's Theorem to the restricted representation of $\mathcal{Z}$ or $\mathcal{D}$, respectively, to obtain a one dimension invariant subspace. With that in mind, the section begins with the following easily overlooked technical difficulty.

Lemma 5.3.1. *Let G be any group. Then there exists an irreducible invariant subspace*

$$W \leq V$$

for every finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$.

Proof. If V is not already irreducible, then find an G-submodule $W \leq V$. Repeat this argument with W to find another submodule $W_1 \leq W$. Finally, V is finite dimensional. Therefore inductively, one can find the desired irreducible. $\square$

For Definitions 5.3.2 and 5.3.4, and Theorem 5.3.3, let G be equal to $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$, and let $\mathcal{H}$ equal $\mathcal{Z}$ or $\mathcal{D}$, respectively. Finally, set $\mathcal{C}_\pm$ to $\mathcal{A}_\pm$ or $\mathcal{B}_\pm$, respectively.

Definition 5.3.2. A *weight* and its corresponding *weight space* for a complex analytic representation $\rho : G \to \mathrm{GL}(V)$ is a pair (μ, V_μ) is a simultaneous eigenvalue/space pair of the representation $\rho|_\mathcal{H} : \mathcal{H} \to \mathrm{GL}(V)$ of $\mathcal{H}$.

Furthermore, a *weight vector* v is any non-zero element of a weight space.

For the following, consider a representation $\rho : G \to \mathrm{GL}(V)$, and apply Lie's theorem and Lemma 5.3.1 to the restriction $\rho|_{\mathcal{C}_+}$ to get a simultaneous eigenvalue/eigenvector pair (μ, v) for $\mathcal{C}_+$.

Theorem 5.3.3. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then $v \in V$ is a simultaneous eigenvector of the restriction $\rho|_{\mathcal{C}_+}$ if and only if v is a weight vector of ρ invariant under the action of $\mathcal{U}$, i.e.*

$$\rho(U)v = v$$

for all $U \in \mathcal{U}$.

Proof. First, if $v \in V$ is simultaneous eigenvector of the restriction $\rho|_{\mathcal{C}_+}$, then clearly v is weight vector of ρ since $\mathcal{H}$ is a subgroup of $\mathcal{C}_+$

Conversely, suppose $v \in V$ is a weight vector of ρ invariant under the action of $\mathcal{U}$. Let μ denote the corresponding weight. Now, take $C \in \mathcal{C}_+$. Then, for some $D \in \mathcal{H}$ and $U \in \mathcal{U}$,

$$C = DU.$$

Thus,

$$\begin{aligned} \rho(C)v &= (\rho(D)\rho(U))v \\ &= \rho(D)v \\ &= \mu(D)v. \end{aligned}$$

Therefore, $v \in V$ is simultaneous eigenvector of the restriction $\rho|_{\mathcal{C}_+}$. $\square$

Definition 5.3.4. Any such $v \in V$ satisfying Theorem 5.3.3 is a *highest (maximal) weight vector*, and its defining weight μ is a *highest (maximal) weight.* Furthermore, considering Theorem 5.3.3 in the case of $\mathcal{C}_-$, one defines a *lowest (minimal) weight vector* and a corresponding *lowest (minimal) weight.*

For simplicity, the remaining results of this section will be established using just complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$. However, know that these results will apply analogously to complex analytic representations of $\mathrm{GL}(n, \mathbb{C})$. For clarification purposes, the section will close on a remark concerning the case of $\mathrm{GL}(n, \mathbb{C})$.

Now that weights of a complex analytic representation of $\mathrm{SL}(n, \mathbb{C})$ have been introduced, what remains is identifying irreducibles by their highest weights. Up until now it has just been stated as fact, but how exactly is it possible that an irreducible is identifiable by a highest weight? In theory, it could be that there exists multiple highest weights to one irreducible, and as consequence, would be the compromise of the utility of unique identification using highest weights. Fortunately, the reality is each irreducible representation has only one highest weight space, which is also one dimensional. With that said, this section justifies this claim and will conclude by showing that if two irreducible representations share a common highest weight, then they must be equivalent.

To begin, recall the definition of dual module from Section 1.5. Let $C(\mathrm{SL}(n,\mathbb{C}))$ denote the set of continuous complex-valued functions on $\mathrm{SL}(n,\mathbb{C})$. Additionally, for V an irreducible $\mathrm{SL}(n,\mathbb{C})$-module, let

$$f_\alpha : V \to C(\mathrm{SL}(n,\mathbb{C}))$$

be the function from Lemma 1.5.5 adapted to the case of $C(\mathrm{SL}(n,\mathbb{C}))$. For a fixed $\alpha \in V^*$, by Lemma 1.5.5,

$$\langle f_\alpha(v) \mid v \in V \rangle$$

is irreducible. Thus, it is generated, as an $\mathrm{SL}(n,\mathbb{C})$-module, by any nonzero $v \in V$. Therefore, a natural choice is to use any highest weight vector as generator.

Proposition 5.3.5. *Let v be a highest weight vector to the $\mathrm{SL}(n,\mathbb{C})-$module V, and α be a lowest weight vector to the dual $\mathrm{SL}(n,\mathbb{C})-$module V^*. Then the map*

$$f_\alpha(v) : G \to \mathbb{C}$$

has a nonzero restriction to both $\mathcal{LDU}$ and $\mathcal{UDL}$.

Proof. To begin, note the following needed observation. Let $v \in V$, and $T \in \mathrm{Hom}_{\mathbb{C}}(V,V)$. Since $\dim V$ is finite, the euclidean norm and the operator norm are equivalent on $T \in \mathrm{Hom}_{\mathbb{C}}(V,V)$. Thus one can verify that the assignment,

$$T \to T(v),$$

defines a continuous map from $\mathrm{Hom}_{\mathbb{C}}(V,V)$ into V. Therefore $f_\alpha(v)$ is continuous on $\mathrm{SL}(n,\mathbb{C})$.

With that said, the function $f_\alpha(v)$ is also nonzero on $\mathrm{SL}(n,\mathbb{C})$. Indeed, by Lemma 1.5.5, it is true that $f_\alpha(v) \neq 0$ since, as a highest weight vector, $v \neq 0$. Furthermore, by Theorem 5.2.8, $\mathcal{LDU}$ and $\mathcal{UDL}$ are both dense in $\mathrm{SL}(n,\mathbb{C})$. So by continuity, if the respective restrictions of $f_\alpha(v)$ were zero, then $f_\alpha(v)$ itself would be zero. Therefore, the statement is valid. $\square$

Remarkably, it turns out that if $f_\alpha(v)$ is created out of a minimal $\alpha \in V^*$ and a maximal $v \in V$, then one gets the following proposition, which illustrates a close relationship between the corresponding highest weight and lowest weight, with the injective $\mathrm{SL}(n,\mathbb{C})$-homomorphism, $f_\alpha(v)$. Ultimately, this result is why there is exactly one highest weight space, and furthermore, why its dimensional can be no more than one.

Proposition 5.3.6. *Suppose (v,μ) is maximal in the $\mathrm{SL}(n,\mathbb{C})-$module V, and suppose (α,ν) is minimal in the dual $\mathrm{SL}(n,\mathbb{C})-$module V^*. Then, for any $UDL \in \mathcal{UDL}$,*

(1) $f_\alpha(v)(UDL) = \mu(D)^{-1}\alpha(v)$, *alternatively*

(2) $f_\alpha(v)(UDL) = \nu(D)\alpha(v)$, *and*

(3) $\alpha(v) \neq 0$

In particular,

$$\nu(D) = \mu(D)^{-1}$$

for all $D \in \mathcal{D}$.

Proof. First, let ρ and ρ^* denote the representations carried by V and V^*, respectively. By hypothesis,

$$\rho^*(DL)\alpha = \nu(D)\alpha,$$

and

$$\rho(UD)^{-1}v = \mu(D)^{-1}v.$$

Thus

$$f_\alpha(v)(UDL) = \alpha(\rho(L)^{-1}\rho(UD)^{-1}v) = \rho^*(L)(\alpha)(\mu(D)^{-1}v) = \mu(D)^{-1}\alpha(v).$$

Alternatively,

$$f_\alpha(v)(UDL) = \alpha(\rho(DL)^{-1}\rho(U)^{-1}v) = \rho^*(DL)(\alpha)(v) = \rho^*(D)(\alpha)(v) = \nu(D)\alpha(v).$$

Now suppose $\alpha(v) = 0$. Then

$$f_\alpha(v)(UDL) = \mu(D^{-1})\alpha(v) = 0.$$

Therefore $f_\alpha(v)$ is the zero map on $\mathcal{UDL}$, contradicting Proposition 5.3.5. $\square$

From Proposition 5.3.6, one may find a highest weight vector in V such that $\alpha(v) = 1$. As a result,

$$f_\alpha(v)(UDL) = \mu(D^{-1})$$

for any $UDL \in \mathcal{UDL}$. This observation will come into play in the following theorem.

Theorem 5.3.7. *There is a unique one dimensional highest weight space of the irreducible* $\mathrm{SL}(n, \mathbb{C})-$*module* V.

Proof. Suppose (v, μ) and (w, η) are both maximal in V, and suppose (α, ν) is minimal in V^* such that $\alpha(v) = 1$. First, $f_\alpha(w) = \alpha(w)f_\alpha(v)$ as functions on $\mathrm{SL}(n, \mathbb{C})$. Indeed, by Proposition 5.3.6,for any $UDL \in \mathcal{UDL}$,

$$f_\alpha(v)(UDL) = \nu(D) = \frac{f_\alpha(w)(UDL)}{\alpha(w)}.$$

Thus, the two continuous functions $f_\alpha(w)$ and $\alpha(w)f_\alpha(v)$ agree on the dense subset $\mathcal{UDL}$. Therefore, they must agree on all of $\mathrm{SL}(n, \mathbb{C})$.

Now by Lemma 1.5.5, $f_\alpha : V \to C(\mathrm{SL}(n,\mathbb{C}))$ is an injective $\mathrm{SL}(n,\mathbb{C})-$homomorphism. Thus $\alpha(w)f_\alpha(v) = f_\alpha(\alpha(w)v)$, and hence,

$$w = \alpha(w)v.$$

Therefore, by Lemma 5.1.3,

$$\mu = \eta.$$

□

This section concludes by justifying the claim that an irreducible $\mathrm{SL}(n,\mathbb{C})-$module is uniquely determined by its highest weight.

Theorem 5.3.8. *Suppose V and W are irreducible $\mathrm{SL}(n,\mathbb{C})-$modules, and let μ and η be their respective highest weights. Then, V and W are isomorphic if and only if $\mu = \eta$.*

Proof. Let ρ and ϱ be the respective representations carried by the $\mathrm{SL}(n,\mathbb{C})$-modules, V and W.

First $\mu = \eta$, whenever $V \cong W$. Indeed, let $\phi : V \to W$ provide the isomorphism, and suppose v is a maximal weight vector in V. Then, for $B = DU \in \mathcal{B}_+$,

$$\varrho(B)\phi(v) = \phi(\rho(B)v) = \phi(\mu(D)v) = \mu(D)\phi(v).$$

Hence, $(\phi(v), \mu)$ is maximal in W. Therefore, by Theorem 5.3.7, $\mu = \eta$.

Conversely, suppose that (v, μ) and (w, η) are maximal in V and W, respectively, and let (α, ν) and (β, υ) be minimal in V^* and W^*, respectively. Furthermore, assume that $\alpha(v) = \beta(w) = 1$. Now, by Proposition 5.3.6, for $UDL \in \mathcal{UDL}$,

(1) $f_\alpha(v)(UDL) = \mu(D)^{-1}$, and

(2) $f_\beta(w)(UDL) = \eta(D)^{-1}$.

So, if $\mu = \eta$, then $f_\alpha(v)$ and $f_\beta(w)$ agree when restricted to $\mathcal{UDL}$, and as a result, $f_\alpha(v) = f_\beta(w)$. Therefore, by Lemma 1.5.5,

$$V \cong \langle f_\alpha(v)\rangle \cong \langle f_\beta(w)\rangle \cong W.$$

□

Remark. By making the following substitutions,

$$\begin{array}{ccc} \mathcal{D} & \to & \mathcal{Z} \\ \mathcal{B}_+ & \to & \mathcal{A}_+ \\ \mathcal{B}_- & \to & \mathcal{A}_-. \end{array}$$

one sees the same results out of this section for the case of $\mathrm{GL}(n,\mathbb{C})$. Most importantly, one has that the irreducible $\mathrm{GL}(n,\mathbb{C})$-modules are also uniquely determined by their highest weights. Ultimately, this is a consequence of the fact that Lie's Theorem applies to $\mathcal{Z}$, $\mathcal{A}_+$, and $\mathcal{A}_-$; and in addition, since $\mathrm{GL}(n,\mathbb{C})$ possess the dense subsets, $\mathcal{LZU}$ and $\mathcal{UZL}$.

5.4 Description of weights

The final goal of this chapter is to show that if μ is a highest weight for an irreducible module of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$, then μ is determined by a sequence of weakly increasing integers,

$$(m_1, m_2, ..., m_n).$$

However, for the case of $\mathrm{SL}(n, \mathbb{C})$, the integers in sequence have the additional property of being nonnegative. In other words, highest weight for an irreducible module of $\mathrm{SL}(n, \mathbb{C})$ are determined always by some integer partition of some appropriate positive integer!

5.4.1 Weights as analytic homomorphisms

To begin, the following lemma is extremely important. Indeed, it shows how the complex analytic nature of the representations result in the fact that weights, in general, must be determined by a sequence of integers.

Lemma 5.4.1. *Let $n \geq 1$, and let $\mu : (\mathbb{C}^*)^n \to \mathbb{C}^*$ be an analytic homomorphism, then μ is determined by a sequence of integers*

$$(m_1, m_2, ..., m_n).$$

That is, for all $\mathbf{z} = (z_1, z_2, ..., z_n)$,

$$\mu(\mathbf{z}) = z_1^{m_1} z_2^{m_2} ... z_n^{m_n}.$$

Proof. First, it will be shown that the analytic homomorphism $\mu : \mathbb{C}^* \to \mathbb{C}^*$ must have the form

$$\mu(z) = z^m$$

for some integer m. From there, the result will extended to the nth direct product, $(\mathbb{C}^*)^n$. By doing so, the proof will be complete.

To begin, recall that exponentiation by the complex number c on $\mathbb{C}^*$ can be defined by

$$w^c := \mathrm{e}^{c \operatorname{Log} w}$$

where $\operatorname{Log} w := \ln |w| + i \operatorname{Arg} w$ is the function defined using the principal value of the complex logarithm and the principal value of the complex argument. Now for nonzero complex numbers $w_1$and w_2 the following holds

$$\begin{aligned}
(w_1 w_2)^c &= \mathrm{e}^{c \operatorname{Log}(w_1 w_2)} \\
&= \mathrm{e}^{c(\ln |w_1 w_2| + i \operatorname{Arg}(w_1 w_2))} \\
&= \mathrm{e}^{c(\operatorname{Log} w_1 + \operatorname{Log} w_2 + i 2\pi N)}
\end{aligned}$$

where $N \in \{-1, 0, 1\}$ is defined such that $\operatorname{Arg} w_1 w_2 = \operatorname{Arg} w_1 + \operatorname{Arg} w_2 + i2\pi N$ holds. Now, using the fact that $\mathrm{e}^{z+w} = \mathrm{e}^{z}\,\mathrm{e}^{w}$, one has that

$$(w_1 w_2)^c = w_1^c w_2^c\,\mathrm{e}^{ci2\pi N}\,.$$

Furthermore, $\mathrm{e}^{ci2\pi N} = 1$ if and only if $c \in \mathbb{Z}$. Consequently, the assignment $w \to w^c$ is a homomorphism if and only if c is an integer.

Now, let $w \in \mathbb{C}^*$ and notice that if μ is an analytic homomorphism on $\mathbb{C}^*$, then

$$\mu'(w) = \mu'(1)\frac{\mu(w)}{w}.$$

Indeed, if $w \neq 0$, then for any $z \in \mathbb{C}^*$,

$$\mu(z) = \mu\left(\frac{z}{w}\right)\mu(w).$$

Therefore,

$$\begin{aligned}
\mu'(w) &= \lim_{z\to w}\frac{\mu(z)-\mu(w)}{z-w} \\
&= \lim_{z\to w}\frac{\mu(\frac{z}{w})-\mu(1)}{\frac{z}{w}-1}\cdot\frac{\mu(w)}{w} \\
&= \mu'(1)\frac{\mu(w)}{w}.
\end{aligned}$$

Realize that this result restricts the possibility for μ by condition that it must be the unique solution to the complex initial value problem

$$\begin{aligned}
\mu'(w) &= c\frac{\mu(w)}{w} \\
\mu(1) &= 1
\end{aligned}$$

on the open set $\mathbb{C}^*$. Indeed, from complex analysis is the following: If $U \subseteq \mathbb{C}$ is a connected open set such that the function $g : U \to \mathbb{C}$ has an anti-derivative, then the anti derivative for g is unique up to constant. Moreover, that constant will be determined by the provided initial condition.

With this consider,

$$\begin{aligned}
g(w) &= w^c \\
&= \mathrm{e}^{c\,\mathrm{Log}\,w},
\end{aligned}$$

for some non-integer $c \in \mathbb{C}$. Now, $f(w) = \mathrm{Log}\,w$ is differentiable everywhere except the on non-positive reals, since it satisfies the polar form of Cauchy-Riemann equations there. With this, one sees that the same is true for g. So, by the chain rule, one has

$$\begin{aligned} g'(w) &= c\,\mathrm{e}^{c\,\mathrm{Log}\,w}\,f'(w) \\ &= c\frac{\mathrm{e}^{c\,\mathrm{Log}\,w}}{w} \\ &= c\frac{g(w)}{w}, \end{aligned}$$

valid on the open connected set $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. And, since $g(1) = 1$, one has $g'(1) = c$. Consequently, the only function h satisfying the complex differential equation $h'(w) = c\frac{h(w)}{w}$ with the initial condition $h(1) = 1$ on U is

$$h(w) = w^c,$$

where c can now be any complex number. Therefore, the analytic homomorphism $\mu : \mathbb{C}^* \to \mathbb{C}^*$ must be of the form

$$\mu(w) = w^c,$$

for some complex number c. However, it was already shown that for a function like this to be a homomorphism it must be true that $c \in \mathbb{Z}$.

Now, let $n \geq 1$, and suppose $\mu : (\mathbb{C}^*)^n \to \mathbb{C}^*$ is an analytic homomorphism. First, for $j \in [n]$, define the function

$$\begin{aligned} \mu_j &: \mathbb{C}^* \to \mathbb{C}^* \\ &: \mathbf{z} \to \mu((z)_j), \end{aligned}$$

where $(z)_j \in (\mathbb{C}^*)^n$ is defined to be the the n-tuple with the jth component equal to the complex number z and every other component equal to 1. It is not too hard to see that μ_j is an analytic homomorphism on $\mathbb{C}^*$. As a result, for all $j \in [n]$, there is some $m_j \in \mathbb{Z}$ such that

$$\mu((z)_j) = z^{m_j}$$

for all elements of this form. So let $\mathbf{z} \in (\mathbb{C}^*)^n$, and realize $\mathbf{z}$ decomposes as

$$\mathbf{z} = (z_1)_1(z_2)_2...(z_n)_n$$

for some set of complex numbers $\{z_1, z_2, ...z_n\}$. Finally, since μ is an analytic homomorphism,

$$\begin{aligned} \mu(\mathbf{z}) &= \mu((z_1)_1(z_2)_2...(z_n)_n) \\ &= \mu((z_1)_1)\mu((z_2)_2)...\mu((z_n)_n) \\ &= z_1^{m_1}z_2^{m_2}...z_n^{m_n}. \end{aligned}$$

Therefore, every analytic homomorphism $\mu : (\mathbb{C}^*)^n \to \mathbb{C}^*$ is given by

$$\mu(\mathbf{z}) = z_1^{m_1}z_2^{m_2}...z_n^{m_n}$$

for some sequence of integers, $(m_1, m_2, ..., m_n)$. $\square$

Theorem 5.4.2. *Let V carry the representation $\rho : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$, and let μ be a weight. Then, for each $D = \mathrm{diag}(d_1, d_2, ..., d_n) \in \mathcal{Z}$,*

$$\mu(D) = d_1^{m_1} d_2^{m_2} ... d_n^{m_n}$$

for some sequence of integers, $(m_1, m_2, ..., m_n)$.

Proof. First note that the restriction $\rho|_{\mathcal{Z}}$ inherits the complex analyticity from ρ. Indeed, the Lie algebra for $\mathcal{Z}$ is the set of diagonal matrices with complex entries. Hence, $\mathcal{Z}$ is a complex Matrix Lie group. So to see that $\rho|_{\mathcal{Z}}$ is complex analytic, apply Corollary 4.4.4 to the induced Lie algebra representation of $\rho|_{\mathcal{Z}}$. Furthermore, with a little more analysis using coordinate projections with some weight vector, one can quickly establish that μ itself complex analytic as well. Finally, $\mathcal{Z}$ clearly has global coordinates given by

$$D \to (d_1, d_2, ..., d_n) \in (\mathbb{C}^*)^n.$$

Therefore, to complete the verification of the claim, one may apply Lemma 5.4.1 to the weight, μ. □

Theorem 5.4.3. *Let V carry the representation $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V)$, and let μ be a weight. Then, for each $D = \mathrm{diag}(d_1, d_2, ..., d_{n-1}, d_n) \in \mathcal{D}$,*

$$\mu(D) = d_1^{m_1} d_2^{m_2} ... d_{n-1}^{m_{n-1}}$$

for some sequence of integers, $(m_1, m_2, ..., m_{n-1})$.

Proof. The proof follows analogously to the proof of Theorem 5.4.2. However, $\mathcal{D}$ has global coordinates given by

$$D \to (d_1, d_2, ..., d_{n-1}) \in (\mathbb{C}^*)^{n-1}$$

as a result of the condition that $\det D = 1$. Therefore, by Lemma 5.4.1, the weight μ will instead be determined by some sequence of integers $(m_1, m_2, ..., m_{n-1})$. □

Remark. Now in some cases, it will be beneficial not single out the last diagonal entry for $D \in \mathcal{D}$ when considering

$$\mu(D).$$

The main reason is that it will be convenient later to identify the weight, μ, with an n-tuple of integers. As a consequence μ will determined some family of integer sequences

$$\{(m_1, m_2, ..., m_{n-1}, 0) + (a, a..., a) \mid a \in \mathbb{Z}\}.$$

To explain, let $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ be a representation and let μ be a weight, with

$$(m_1, m_2, ..., m_n) \in \mathbb{Z}^n.$$

Suppose

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in \mathcal{D}.$$

Then, $d_n = (d_1 d_2 ... d_{n-1})^{-1}$ since $\det D = 1$. With this in mind, one has

$$d_1^{m_1} d_2^{m_2} ... d_{n-1}^{m_{n-1}} (d_1 d_2 ... d_{n-1})^{-m_n} = d_1^{(m_1 - m_n)} d_2^{(m_2 - m_n)} ... d_{n-1}^{(m_{n-1} - m_n)}.$$

Therefore, considering Theorem 5.4.3, μ would be determined by

$$(m_1 - m_n, m_2 - m_n, ..., m_{n-1} - m_n)$$

given $(m_1, m_2, ..., m_n)$.

However, consider a fixed $a \in \mathbb{Z}$. Then,

$$d_1^{(m_1+a)} d_2^{(m_2+a)} ... d_n^{(m_n+a)} = d_1^{m_1} d_2^{m_2} ... d_n^{m_n} (d_1 d_2 ... d_n)^a = d_1^{m_1} d_2^{m_2} ... d_n^{m_n}.$$

Therefore, μ can be associated to $(m_1 + a, m_2 + a, ..., m_n + a)$ for all $a \in \mathbb{Z}$.

Finally, for both cases concerning weights of complex analytic representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, the notation

$$\mu \equiv (m_1, m_2, ..., m_n)$$

will be used to signify that μ is determined by the integer sequence, $(m_1, m_2, ..., m_n)$. The only difference will be whether or not the correspondence is unique.

5.4.2 Weights and weight spaces for $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$

One will need assistance from the Lie algebras $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$ in order to show that if a particular weight is given by a weakly increasing sequence of integers, then it must be the highest weight.

Definition 5.4.4. Let G be equal to $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$. The *adjoint representation*,

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

is defined by setting, for each $X \in \mathfrak{g}$ and $A \in G$,

$$\mathrm{Ad}(A)X := AXA^{-1}.$$

Proposition 5.4.5. *Let G be equal to $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{SL}(n, \mathbb{C})$. Then, the adjoint representation induces a Lie algebra representation of $\mathfrak{g}$ on itself*

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

Explicitly, for each $X, Y \in \mathfrak{g}$,

$$\mathrm{ad}(Y)X = [Y, X].$$

Proof. Let $Y, X \in \mathfrak{g}$. Then,

$$[Y, X] = \frac{d}{dt}\left(\mathrm{e}^{tY}\, X\, \mathrm{e}^{-tY}\right)\Big|_{t=0}.$$

Therefore, by Proposition 4.2.4, the result follows. $\square$

At this point and until the close of this section, results and definitions will be presented and established using just the setting of $\mathrm{SL}(n, \mathbb{C})$ and $\mathfrak{sl}(n, \mathbb{C})$. Again, the motivation is to promote simplicity since, like before, the results extend easily and analogously to the setting of $\mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{C})$. Finally, a closing remark will be given concerning $\mathrm{GL}(n, \mathbb{C})$ for clarification purposes.

Now, the following Lie sub-algebras in $\mathfrak{sl}(n, \mathbb{C})$ are of interest and will be needed.

$$\begin{aligned}
\mathfrak{h} &= \{H \in \mathfrak{sl}(n, \mathbb{C}) \mid H = \mathrm{diag}(h_1, h_2, ..., h_n)\} \\
\mathfrak{u} &= \{X \in \mathfrak{sl}(n, \mathbb{C}) \mid X \text{ is strictly upper triangular}\} \\
\mathfrak{l} &= \{Y \in \mathfrak{sl}(n, \mathbb{C}) \mid Y \text{ is strictly lower triangular}\}
\end{aligned}$$

Recall the basis for $\mathfrak{sl}(n, \mathbb{C})$ introduced in Chapter 4,

$$\mathcal{E} = \{E_{ij} \mid 1 \le i \ne j \le n\} \cup \{H_i \mid i \in [n-1]\}.$$

Using this basis, one can utilize the following convenient formula for the matrix commutator,

$$[E_{ij}, E_{lk}] = \delta_{jl} E_{ik} - \delta_{ki} E_{lj}.$$

Lastly, note the following decomposition

$$\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{h} \oplus \mathfrak{u} \oplus \mathfrak{l}.$$

Here is the notion of weight and weight space seen from the Lie algebra.

Definition 5.4.6. A *weight* and its corresponding *weight space* for the Lie algebra representation $p : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$ is a pair (α, V_α) such that

(1) $\alpha \in \mathfrak{h}^*$, and

(2) $V_\alpha = \{v \in V \mid p(H)v = \alpha(H)v\, \forall H \in \mathfrak{h}\}$ is a non-trivial subspace in V.

Furthermore, a *weight vector* is any non-zero element of a weight space.

Example 5.4.7. Consider $\mathrm{Ad} : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(\mathfrak{sl}(n, \mathbb{C}))$, and the induced lie algebra representation $\mathrm{ad} : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(\mathfrak{sl}(n, \mathbb{C}))$. Let $1 \le l, k \le n$, and

$$H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n \end{bmatrix} \in \mathfrak{h}.$$

Then, one obtains

$$[H, E_{lk}] = (h_l - h_k)E_{lk}.$$

Indeed, write $H = \sum_{i=1}^{n} h_i E_{ii}$. Thus,

$$[H, E_{lk}] = \sum_{i=1}^{n} h_i[E_{ii}, E_{lk}] = \sum_{i=1}^{n} h_i(\delta_{il}E_{ik} - \delta_{ki}E_{li}) = (h_l - h_k)E_{lk}.$$

Therefore, for each $l \neq k$, one has that E_{lk} is weight vector of corresponding weight $\alpha_{lk} \in \mathfrak{h}^*$ given by

$$\alpha_{lk}(H) = h_l - h_k.$$

Finally, H_i is also weight vector for each $i \in [n-1]$ with the zero functional as weight.

The weight vectors of the adjoint representation play an important role in the general theory of Lie algebra representations. Therefore, weight vectors that do not belong to $\mathfrak{h}$ are called *root vectors*, and their corresponding weights are called *roots*. Consider the following.

Theorem 5.4.8. *Let $p : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$ be a representation, and suppose (v, α) is a weight/weight vector pair for p. Then, for each $1 \leq l \neq k \leq n$,*

(1) *$p(E_{lk})v$ is another weight with corresponding weight $\alpha + \alpha_{lk}$, or*

(2) *$p(E_{lk})v = 0$.*

Proof. Let $H \in \mathfrak{h}$. Recall that

$$p([H, E_{lk}])v = p(H)(p(E_{lk})v) - p(E_{lk})(p(H)v).$$

Thus,

$$\begin{aligned} p(H)(p(E_{lk})v) &= p([H, E_{lk}])v + p(E_{lk})(p(H)v) \\ &= (\alpha_{lk}(H) + \alpha(H))p(E_{lk})v. \end{aligned}$$

Therefore, if $p(E_{lk})v \neq 0$, then $p(E_{lk})v$ is another weight vector with weight $\alpha + \alpha_{lk}$. □

Proposition 5.4.9. *Let $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ be a representation for $\mathrm{SL}(n, \mathbb{C})$, and let $\dot{\rho} : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$ be the induced Lie algebra representation for $\mathfrak{sl}(n, \mathbb{C})$. Then, v is a weight vector for $\rho|_{\mathcal{D}}$ if and only if v is a weight vector for $\dot{\rho}$.*

Furthermore, suppose μ and α are the corresponding respective weights for $\rho|_{\mathcal{D}}$ and $\dot{\rho}$ such that

$$\mu \equiv (m_1, m_2, ..., m_n) \in \mathbb{Z}^n.$$

Then, for all $H \in \mathfrak{h}$,

$$\alpha(H) = m_1h_1 + m_2h_2 + ... + m_nh_n.$$

Proof. Suppose $v \in V$ is a weight vector for $\rho|_{\mathcal{D}}$ with weight $\mu \equiv (m_1, m_2, ..., m_n)$, and let

$$H = \begin{bmatrix} h_1 & 0 & \cdots & 0 \\ 0 & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_n \end{bmatrix} \in \mathfrak{h}.$$

Using Proposition 4.2.4,

$$\begin{aligned} \dot{\rho}(H)v &= \left(\lim_{t \to 0} \frac{\rho(\mathrm{e}^{tH}) - \mathrm{id}_V}{t} \right) v \\ &= \lim_{t \to 0} \frac{\mu(\mathrm{e}^{tH})v - v}{t} \\ &= \left(\lim_{t \to 0} \frac{\mu(\mathrm{e}^{tH}) - 1}{t} \right) v \\ &= \left(\frac{d}{dt} \left(\mu(\mathrm{e}^{tH})\right)\Big|_{t=0} \right) v. \end{aligned}$$

Now,

$$\mathrm{e}^{tH} = \begin{bmatrix} \mathrm{e}^{h_1 t} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{h_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{h_n t} \end{bmatrix}.$$

Thus,

$$\begin{aligned} \mu(\mathrm{e}^{tH}) &= (\mathrm{e}^{h_1 t})^{m_1} (\mathrm{e}^{h_2 t})^{m_2} ... (\mathrm{e}^{h_n t})^{m_n} \\ &= \mathrm{e}^{(m_1 h_1 + m_2 h_2 + ... + m_n h_n)t}. \end{aligned}$$

With this,

$$\frac{d}{dt} \left(\mu(\mathrm{e}^{tH})\right)\Big|_{t=0} = m_1 h_1 + m_2 h_2 + ... + m_n h_n.$$

Therefore, since

$$\dot{\rho}(H)v = (m_1 h_1 + m_2 h_2 + ... + m_n h_n)v,$$

v is a weight vector for $\dot{\rho}$ with weight α, given by

$$\alpha(H) = m_1 h_1 + m_2 h_2 + ... + m_n h_n.$$

Conversely, suppose that v is a weight vector for $\dot{\rho}$ with weight α, and let

$$D = \begin{bmatrix} \delta_1 & 0 & \cdots & 0 \\ 0 & \delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_n \end{bmatrix} \in \mathcal{D}.$$

First, for each $i \in [n-1]$, pick a $z_i \in \mathbb{C}$ such that $\delta_i = \mathrm{e}^{z_i}$, since $\delta_i \neq 0$. Realizing that $\delta_n = (\delta_1 \delta_2 ... \delta_{n-1})^{-1}$, set $z_n = -(z_1 + z_2 + ... + z_{n-1})$. With this in mind, define

$$H = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix}.$$

As a result, one has $H \in \mathfrak{h}$, and $\mathrm{e}^H = D$. Now using Proposition 4.2.4, one can show

$$\begin{aligned} \rho(D)v &= \mathrm{e}^{\dot{\rho}(H)}\, v \\ &= \mathrm{e}^{\alpha(H)}\, v, \end{aligned}$$

since

$$\left(\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\dot{\rho}(H)^k}{k!}\right) v = \lim_{k\to\infty} \left(\sum_{i=1}^{k} \frac{\alpha(H)^k}{k!} v\right) = \left(\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\alpha(H)^k}{k!}\right) v.$$

Therefore, v is weight vector for $\rho|_{\mathcal{D}}$ for some weight μ given by

$$\mu(D) = \mathrm{e}^{\alpha(H)}$$

such that $H \in \mathfrak{h}$ satisfies $\mathrm{e}^H = D$. □

Remark. Considering 5.4.9, $(m_1, m_2, ..., m_n)$ can represent both weights μ and α. Also, one can simply use the term 'weight vector' or 'weight' without having to specify ρ or $\dot{\rho}$.

5.4.3 The lexicographic order on weights and permutation matrices

Now, the final step is showing that if a weight is a highest weight, then it is given by an increasing sequence of integers. This will be accomplished by the use of the lexicographic order, and by the additional aid of permutation matrices. To start, in order to define the lexicographic order one will need a unique representative $(m_1, m_2, ..., m_n)$ for a given weight μ. So considering Theorem 5.4.3, choose $(m_1, m_2, ..., m_n)$ such that $m_n = 0$.

Definition 5.4.10. Suppose $\mu \equiv (m_1, m_2, ..., m_{n-1}, 0)$ and $\nu \equiv (k_1, k_2, ..., k_{n-1}, 0)$ are two weights, of an irreducible $\mathrm{SL}(n, \mathbb{C})$- module V. Then, $\mu \geq \nu$ in the *lexicographic order*, whenever,for some $i \in [n]$,

$$m_j = k_j \text{ for all } j < i, \text{ and } m_i \geq k_i.$$

Note that this a total order on weights with a maximal element since, by Lemma 5.1.3, there are only finitely many distinct weights. In fact, one has the following useful result.

Lemma 5.4.11. *Let $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ be an irreducible representation for $\mathrm{SL}(n, \mathbb{C})$, and let μ be a weight. Then, μ is maximal with respect to the lexicographic order, if and only if μ is the highest weight.*

Proof. In this proof, one appeals to the induced representation $\dot{\rho} : \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(V)$.

Let v be a weight vector, and suppose $\mu, \alpha \equiv (m_1, m_2, ..., m_{n-1}, m_n)$, where $m_n = 0$. (Recall Proposition 5.4.9.)

Now suppose μ is maximal with respect to the lexicographic order, and let $1 \leq i < j \leq n$. Then, in consideration of Theorem 5.4.8,

$$\dot{\rho}(E_{ij})v$$

is another weight vector with weight $\alpha + \alpha_{ij}$, whenever $\dot{\rho}(E_{ij})v \neq 0$. So, take

$$H = \begin{bmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & h_n \end{bmatrix} \in \mathfrak{h},$$

and note that

$$\begin{aligned} (\alpha + \alpha_{ij})(H) &= \alpha(H) + \alpha_{ij}(H) \\ &= \sum_{l=1}^{n} m_l h_l + (h_i - h_j) \\ &= (m_i + 1)h_i + (m_j - 1)h_j + \sum_{l \neq i,j} h_l m_l. \end{aligned}$$

Consequently, if $\dot{\rho}(E_{ij})v$ is another weight vector with $1 \leq i < j \leq$, then there exists a weight

$$\nu \equiv (m_1, ..., m_i + 1, ..., m_j - 1, ..., m_n).$$

However, this would imply that $\nu > \mu$, with respect to the lexicographic order. Thus, under the assumption that μ is maximal, it must be that, for all $1 \leq i < j \leq n$,

$$\dot{\rho}(E_{ij})v = 0.$$

As a result, v is annihilated by all of $\mathfrak{u}$.

Now, this implies that v is fixed by all of $\mathcal{U}$. Indeed, $\dot{\rho}(X)v = 0$ for all $X \in \mathfrak{u}$, if and only if $\rho(U)v = v$ for all $U \in \mathcal{U}$. Hence, by Theorem 5.3.3, v is a highest weight vector, and therefore μ is the highest weight.

Alternatively, suppose μ is not maximal with respect to the lexicographic order. Then, there is some other weight $\nu > \mu$ that is maximal. But considering the previous argument, ν is then highest weight. Therefore, by Theorem 5.3.7, μ is not the highest weight. $\square$

Like the lexicographic order, permutation matrices play a key role. The utility of such matrices is illustrated by the following.

Lemma 5.4.12. *Let* $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(V)$ *be a representation. Suppose* $(m_1, m_2, ..., m_n)$

defines a weight. Then, for any permutation $\sigma \in \mathcal{S}_n$,

$$(m_{\sigma(1)}, m_{\sigma(2)}, ..., m_{\sigma(n)})$$

defines a weight as well.

Proof. The following matrix will be utilized in this proof;

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

With that said, recall that for a given $\sigma \in \mathcal{S}_n$, one can define a *permutation* matrix by permuting the columns of the identity matrix with σ. So, let P_σ be that permutation matrix. Then,

$$P_\sigma = \sum_{i=1}^{n} E_{i\sigma(i)},$$

which has the property that

$$\det(P_\sigma) = \mathrm{sgn}(\sigma).$$

Indeed,

$$\det : \mathbb{C}^n \to \mathbb{C}$$

is completely anti-symmetric.

Now, realize that $P_\sigma \in \mathrm{SL}(n, \mathbb{C})$ if and only if σ is an even permutation. Therefore, U will act as a correction term since UP_σ will be in $\mathrm{SL}(n, \mathbb{C})$. In other words,

$$\det(UP_\sigma) = \det(U)\mathrm{sgn}(\sigma) = (-1)^2 = 1.$$

Continuing, Let $D \in \mathcal{D}$, and consider $(P_\sigma)^{-1}DP_\sigma$. Note that $(P_\sigma)^{-1} = P_{\sigma^{-1}}$, and write

$$D = \sum_{i=1}^{n} \delta_i E_{ii}.$$

Then, using the multiplication formula

$$E_{lk}E_{ij} = \delta_{ki}E_{lj},$$

one can easily verify that

$$(P_\sigma)^{-1}DP_\sigma = \sum_{i=1}^{n} \delta_{\sigma^{-1}(i)} E_{ii}.$$

In other words, if

$$D = \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \delta_n \end{bmatrix},$$

then

$$(P_\sigma)^{-1} D P_\sigma = \begin{bmatrix} \delta_{\sigma^{-1}(1)} & 0 & 0 & 0 \\ 0 & \delta_{\sigma^{-1}(2)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \delta_{\sigma^{-1}(n)} \end{bmatrix}.$$

Hence, $(P_\sigma)^{-1} D P_\sigma \in \mathcal{D}$. Additionally,

$$(UP_\sigma)^{-1} D (UP_\sigma) = (P_\sigma)^{-1} D P_\sigma,$$

since $U^{-1} D U = D$.

Now, for convenience set $(P_\sigma)^{-1} D P_\sigma = D_\sigma$, and suppose that $v \in V$ is a weight vector with corresponding weight $\mu \equiv (m_1, m_2, ..., m_n)$. If σ is even, then

$$\rho(D)\rho(P_\sigma)v = \rho(P_\sigma)\rho(P_\sigma)^{-1}\rho(D)\rho(P_\sigma)v = \rho(P_\sigma)\rho(P_\sigma^{-1} D P_\sigma)v,$$

hence,

$$\rho(D)\rho(P_\sigma)v = \mu(D_\sigma)\rho(P_\sigma)v.$$

Consequently, $\rho(P_\sigma)v$ is another weight vector with weight ν, such that $\nu(D) = \mu(D_\sigma)$ for all $D \in \mathcal{D}$. Furthermore,

$$(\delta_{\sigma^{-1}(1)})^{m_1} (\delta_{\sigma^{-1}(2)})^{m_2} ... (\delta_{\sigma^{-1}(n)})^{m_n} = (\delta_1)^{m_{\sigma(1)}} (\delta_2)^{m_{\sigma(2)}} ... (\delta_n)^{m_{\sigma(n)}}.$$

Thus,

$$\nu(D) = (\delta_1)^{m_{\sigma(1)}} (\delta_2)^{m_{\sigma(2)}} ... (\delta_n)^{m_{\sigma(n)}}$$

for all $D \in \mathcal{D}$, and therefore, $\nu \equiv (m_{\sigma(1)}, m_{\sigma(2)}, ..., m_{\sigma(n)})$.

Finally, if σ is odd, then repeat the previous argument with UP_σ, since

$$\rho(UP_\sigma)^{-1}\rho(D)\rho(UP_\sigma) = \rho({P_\sigma}^{-1} D P_\sigma) = \rho(D_\sigma).$$

In doing so, one finds that $\rho(UP_\sigma)v$ is another weight vector with corresponding weight $\nu \equiv (m_{\sigma(1)}, m_{\sigma(2)}, ..., m_{\sigma(n)})$. □

With the establishment of Lemma 5.4.11 and 5.4.12, it is now time to verify the final needed result linking highest weights to weakly increasing sequences of integers.

Theorem 5.4.13. *Suppose $\mu \equiv (m_1, m_2, ..., m_{n-1}, 0)$ is the highest weight for the irreducible* $\mathrm{SL}(n, \mathbb{C})$*- module, V. Then,*

$$m_1 \geq m_2 \geq ... \geq m_{n-1} \geq 0.$$

Proof. Suppose $\mu \equiv (m_1, m_2, ..., m_{n-1}, 0)$ is the unique highest weight for the irreducible

SL$(n, \mathbb{C})$-module V. Then, by Lemma 5.4.11, μ is also the maximal weight with respect to the lexicographic order. Now, if $(m_1, m_2, ..., m_{n-1}, m_n)$, with $m_n = 0$, did not already satisfy the condition

$$m_1 \geq m_2 \geq ... \geq m_{n-1} \geq 0,$$

then, for some appropriate $\sigma \in \mathcal{S}_n$, one could find a different weight

$$\nu \equiv (m_{\sigma(1)}, m_{\sigma(2)}, ..., m_{\sigma(n-1)}, m_{\sigma(n)}),$$

such that

$$m_{\sigma(1)} \geq m_{\sigma(2)} \geq ... \geq m_{\sigma(n-1)} \geq m_{\sigma(n)}.$$

However, $\mu > \nu$, since $\mu \neq \nu$. Thus, for some $i \in [n]$,

$$m_j = m_{\sigma(j)} \text{ for all } j < i, \text{ and } m_i > m_{\sigma(i)}.$$

With this in mind, realize that $i \geq 2$. True, otherwise

$$m_1 > m_{\sigma(1)} \geq m_{\sigma(2)} \geq ... \geq m_{\sigma(n-1)} \geq m_{\sigma(n)},$$

which implies $m_1 > m_1$. Consequently, $m_1 = m_{\sigma(1)}$. And therefore, $m_1 \geq m_l$ for all $l \in [n]$.

Now, one can just assume that $\sigma(1) = 1$ since m_1, being maximal, did not need to be permuted from the start. Moving forward, one has that

$$m_2 \geq m_{\sigma(2)} \geq ... \geq m_{\sigma(n-1)} \geq m_{\sigma(n)},$$

since $i \geq 2$. So, by applying the previous argument again, one finds $m_2 = m_{\sigma(2)}$, and $m_1 \geq m_2 \geq m_l$ for all $l \geq 2$. Hence, like before, it can be assumed that $\sigma(2) = 2$.

By continuing this, one sees

$$m_1 \geq m_2 \geq ... \geq m_{i-1} \geq m_l,$$

for all $l \geq i$, and thus it can be assumed that $\sigma(k) = k$ for $1 \leq k \leq i-1$. However, one still obtains a contradiction. Indeed, σ permutes the last $n - i + 1$ entries of

$$(m_1, m_2, ..., m_{n-1}, 0)$$

amongst themselves, since $\sigma(k) = k$ for $1 \leq k \leq i-1$. Furthermore, $m_i > m_{\sigma(i)}$, implies that $m_i \geq m_{\sigma(l)}$ for all $i \leq l \leq n$. But then, $m_i > m_i$. Therefore

$$m_1 \geq m_2 \geq ... \geq m_{n-1} \geq 0,$$

for the highest weight $\mu \equiv (m_1, m_2, ..., m_{n-1}, 0)$. □

Remark. By repeating the methods presented in this section, but with the following sub-algebras of $\mathfrak{gl}(n,\mathbb{C})$,

$$\begin{aligned}
\mathfrak{z} &= \{H \in \mathfrak{gl}(n,\mathbb{C}) \mid H = \mathrm{diag}(h_1, h_2, ..., h_n)\} \\
\mathfrak{u} &= \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X \text{ is strictly upper triangular}\} \\
\mathfrak{l} &= \{Y \in \mathfrak{gl}(n,\mathbb{C}) \mid Y \text{ is strictly lower triangular, }\}
\end{aligned}$$

one sees the same results for the case of $\mathrm{GL}(n,\mathbb{C})$. However, now the highest weights of irreducible $\mathrm{GL}(n,\mathbb{C})$-modules are uniquely determined by n integers satisfying

$$m_1 \geq m_2 \geq ... \geq m_{n-1} \geq m_n.$$

The main reason for this is that $\mathcal{Z}$ has coordinates consisting of n non-zero complex numbers instead of $n-1$, and that weights are determined uniquely by n integers with no condition being placed on the trailing integer.

In summary, distinct irreducible complex analytic Lie group representations of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$ are determined by their highest weights. For $\mathrm{GL}(n,\mathbb{C})$, these highest weights are themselves determined uniquely by n integers satisfying

$$m_1 \geq m_2 \geq ... \geq m_n.$$

Where as, for $\mathrm{SL}(n,\mathbb{C})$, the highest weights are one to one correspondence with $n-1$ integers having the following property,

$$m_1 \geq m_2 \geq ... \geq m_{n-1} \geq 0.$$

Such sequences were said to be nothing more than integer partitions of an appropriate positive integer.

Now, in order to provide a complete classification of irreducible representations of $\mathrm{SU}(n)$, realizations still need to be provided. Considering Chapter 3, and as suggested previously, the various image spaces of the general projection operators corresponding to integer partitions, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$, provide the source of these realizations. However, it still needs to be established that these are irreducible when seen as modules to $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$. Furthermore, considering this chapter, if one wants to characterize the complete classification of irreducible representations of $\mathrm{SU}(n)$ in terms of integer partitions, then the following question must be answered. Which partitions determine highest weights to irreducible complex analytic representations of $\mathrm{SL}(n,\mathbb{C})$? So with these issues in mind, the following chapter picks up where Chapter 3 left off, in setting of the irreducible tensor representations of $\mathrm{GL}(n,\mathbb{C})$.

Chapter 6

Irreducible representations of $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$

The final step to provide a complete classification of irreducible representations of $\mathrm{SU}(n)$ is to provide realizations in setting of the irreducible tensor representations of $\mathrm{GL}(n, \mathbb{C})$. In this chapter, the accomplishment of this last task will follow by first showing that the nonzero image spaces of the various general projection operators are additionally irreducible $\mathrm{SL}(n, \mathbb{C})$-modules. Afterward, solely in terms of the defining integer partition and the positive integer n, necessary and sufficient conditions for a particular image space of a general projection operator being nontrivial will be provided. From there, given any nontrivial image space of a general projection, the existence of a highest weight vector will be verified, along with the confirmation that the description of the corresponding highest weight is determined exactly by the integer partition associated to the projection operator. Finally, the chapter will conclude with a formal presentation of the classifications of both the finite dimensional irreducible complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$ and finite dimensional irreducible representations of $\mathrm{SU}(n)$.

The exposition presented in this chapter is modeled by the same given by Sternberg [4]. In addition, some needed results also follow from B. Hall [1]. Throughout this chapter, let $V = \mathbb{C}^n$, and $\mathcal{E} = \{e_i \mid i \in [n]\}$ denote the standard basis for V. Lastly, note that all representations are still assumed to be finite.

6.1 The irreducible tensor representations are complex analytic

This section begins by reminding the reader of relevant theory from Chapter 3. Let m be a positive integer. First, recall that both $\mathcal{S}_m$ and $\mathrm{GL}(n, \mathbb{C})$ act on $V^{\bigotimes m}$ via monomials. For $\sigma \in \mathcal{S}_m$, one had

$$\sigma(v_1 \otimes v_2 \otimes ... \otimes v_m) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes ... \otimes v_{\sigma^{-1}(m)}.$$

Likewise,

$$A^{\otimes m}(v_1 \otimes v_2 \otimes ... \otimes v_m) = Av_1 \otimes Av_2 \otimes ... \otimes Av_m,$$

for $A \in \mathrm{GL}(n, \mathbb{C})$. Secondly, recall the following decomposition

$$V^{\bigotimes m} = \bigoplus_{\lambda \vdash m} W^{\lambda}$$

where, for each $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$, W^{λ} is the isotypic component corresponding to the Specht module of shape λ. Furthermore, recall that $U^{\lambda} = \mathrm{Hom}_{\mathcal{S}_m}(\mathcal{S}^{\lambda}, V^{\bigotimes m})$, with $m_{\lambda} = \dim U^{\lambda}$, being the number of isomorphic copies of $\mathcal{S}^{\lambda}$ that appear in the decomposition of $V^{\bigotimes m}$ into irreducible $\mathcal{S}_m$-submodules.

Finally, for each $\lambda \vdash m$ and any tableau t of corresponding shape, one had the following isomorphism of $\mathrm{GL}(n, \mathbb{C})$-modules,

$$U^{\lambda} \cong \epsilon_t \left(V^{\bigotimes m} \right),$$

where ϵ_t denotes the general projection operator built from the row and column stabilizer of t.

Now, in order to apply the results from Chapter 5 to irreducible tensor representations on $V^{\bigotimes m}$, it will appropriate at this time to point out that that the representation

$$T^{\otimes m} : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V^{\bigotimes m})$$

is indeed complex analytic for each m. This fact follows very naturally from the describing action of $\mathrm{GL}(n, \mathbb{C})$, and is verifiable by first showing that the representation

$$T : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V)$$

is complex analytic, and then establishing the claim for each m.

For the first case, one makes the usual identification of $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{C})$ using the map

$$L \to [L]_{\mathcal{E}},$$

where $[L]_{\mathcal{E}}$ denotes the matrix of $L \in \mathrm{GL}(V)$ relative to the standard basis. Under this map, one has $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$ as complex manifolds by virtue of the identification itself! Furthermore, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then, for each $i \in [n]$,

$$TA(e_i) = \sum_{j=1}^{n} a_{ji} e_j.$$

Thus, the matrix entries of $[TA]_{\mathcal{E}}$ are given by the matrix entries of A. Clearly, this representation is complex analytic.

More generally, let $m > 1$. Then

$$\begin{aligned}
A^{\otimes m}(e_{j_1} \otimes e_{j_2} \otimes ... \otimes e_{j_m}) &= Ae_{j_1} \otimes Ae_{j_2} \otimes ... \otimes Ae_{j_m} \\
&= (\sum_{i_1=1}^{n} a_{i_1 j_1} e_{i_1}) \otimes (\sum_{i_2=1}^{n} a_{i_2 j_2} e_{i_2}) \otimes ... \otimes (\sum_{i_m=1}^{n} a_{i_m j_m} e_{i_m}) \\
&= \sum_{i_1, i_2, ..., i_m=1}^{n} a_{i_1 j_1} a_{i_2 j_2} ... a_{i_m j_m} e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}.
\end{aligned}$$

With this in mind, the matrix entries of $A^{\otimes m}$ relative to the basis

$$\{e_{j_1} \otimes e_{j_2} \otimes ... \otimes e_{j_m} \mid (j_1, ..., j_m) \in [n]^m\}$$

are given by products of the entries of A. Therefore, $T^{\otimes m}$ is complex analytic for each m.

Finally consider the following,

$$\mathrm{SU}(n) \subseteq \mathrm{SL}(n, \mathbb{C}) \subseteq \mathrm{GL}(n, \mathbb{C}).$$

So by restriction of $T^{\otimes m}$, one sees that $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ also act on $V^{\bigotimes m}$. Consequently, for each $\lambda \vdash m$, and each tableau t of corresponding shape, the $\mathrm{GL}(n, \mathbb{C})$-module, $\epsilon_t\left(V^{\bigotimes m}\right)$, is additionally a module to both $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$ under the restriction of $T^{\otimes m}$ to each of the respective subgroups. Ultimately it will be shown that as $\mathrm{SL}(n, \mathbb{C})$-module, $\epsilon_t\left(V^{\bigotimes m}\right)$ is irreducible, and therefore, by Theorem 4.4.14, is irreducible as a $\mathrm{SU}(n)$-module as well.

6.2 The standard basis as weight vectors

In order to first show that each $\epsilon_t\left(V^{\bigotimes m}\right)$, carrying the restriction of $T^{\otimes m}$ to $\mathrm{SL}(n, \mathbb{C})$, is still irreducible as an $\mathrm{SL}(n, \mathbb{C})$-module, it will be established that modules that carrying irreducible complex analytic representations of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, in general, are spanned by weight vectors. Afterward in Section 6.3, the desired observation will follow from showing that, for the case of the irreducible tensor representations, weight spaces relative to $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ are, in fact, one and the same. To start, consider the following results borrowed from the work of B. Hall [1].

Definition 6.2.1. Let $p : \mathfrak{g} \to \mathfrak{gl}(W)$ be a Lie algebra representation for $\mathfrak{g}$, and let $w \in W$. Then the *cyclic subspace of* w is

$$\langle w \rangle_p = \langle p(X_1)p(X_2)...p(X_l)w \mid \{X_i\}_{i \in [l]} \subseteq \mathfrak{g},\, l \in \mathbb{N} \rangle.$$

It is straight forward to see that $\langle w \rangle_p$ is a $\mathfrak{g}-$submodule. Indeed, $\langle w \rangle_p$ is defined so that it would be.

Lemma 6.2.2. *Let* $p : \mathfrak{g} \to \mathfrak{gl}(W)$ *be a Lie algebra representation for* $\mathfrak{g}$, *and let*

$$\mathcal{B} = \{X_1, ...X_k\}$$

be an ordered basis of $\mathfrak{g}$, *where* $k = \dim \mathfrak{g}$. *Furthermore, for each* $K \geq 1$, *define the following subspace of* $\mathfrak{gl}(W)$

$$\langle K \rangle = \langle p(X_1)^{m_1} p(X_2)^{m_2} ... p(X_k)^{m_k} \mid \{m_i\}_{i \in [k]} \subseteq \mathbb{N}, \sum_{l=1}^{k} m_l \leq K \rangle.$$

Then, for any $(i_1, ..., i_K) \in [k]^K$,

$$p(X_{i_1}) p(X_{i_2}) ... p(X_{i_K}) \in \langle K \rangle.$$

Proof. This proof will use induction on K. Also, one will need the collection of constants $\{L^i_{jl} \in \mathbb{C} \mid 1 \leq i, j, l \leq k\}$ that define the commutator. In other words, for each $j, l \in [k]$, one has

$$[X_j, X_l] = X_j X_l - X_l X_j = \sum_{i=1}^{k} L^i_{jl} X_i.$$

Now, the claim is obviously true for $K = 1$. So let $K = 2$, suppose $i > j$, and write

$$p(X_i) p(X_j) = [p(X_i), p(X_j)] + p(X_j) p(X_i).$$

Then, since $p([X_i, X_j]) = [p(X_i), p(X_j)]$, one has

$$\begin{aligned} p(X_i) p(X_j) &= p([X_i, X_j]) + p(X_j) p(X_i) \\ &= \sum_{l=1}^{k} L^l_{ij} p(X_l) + p(X_j) p(X_i). \end{aligned}$$

Clearly,

$$\sum_{l=1}^{k} L^l_{ij} p(X_l) \in \langle 2 \rangle,$$

and since $j < i$, one has $p(X_j) p(X_i) \in \langle 2 \rangle$ as well. Thus

$$p(X_i) p(X_j) \in \langle 2 \rangle.$$

Therefore, the result holds for the case $K = 2$.

With this, suppose that for some $N \geq 2$, the claim is true for all $1 \leq K \leq N$. Let $j \in [k]$, and let

$$\{X_j, X_{i_1} X_{i_2} ... X_{i_N}\} \subseteq \mathcal{B}.$$

Then, by the inductive hypothesis, for some $M \geq 1$ and collection of complex numbers

$\{a_l\}_{l\in[M]}$,

$$\begin{aligned} p(X_j)p(X_{i_1})p(X_{i_2})...p(X_{i_N}) &= p(X_j)\left(p(X_{i_1})p(X_{i_2})...p(X_{i_N})\right) \\ &= p(X_j)\left(\sum_{l=1}^{M} a_l p(X_1)^{m_{1l}}p(X_2)^{m_{2l}}...p(X_k)^{m_{kl}}\right) \\ &= \sum_{l=1}^{M} a_l p(X_j)p(X_1)^{m_{1l}}p(X_2)^{m_{2l}}...p(X_k)^{m_{kl}}, \end{aligned}$$

where $m_{il} \geq 0$, and $\sum_{i=1}^{k} m_{il} \leq N$, for each l. Consequently, one only needs to show the result for elements of the form

$$p(X_j)p(X_1)^{m_1}p(X_2)^{m_2}...p(X_k)^{m_k},$$

where $m_i \geq 0$, and $\sum_{l=1}^{k} m_l \leq N$.

Using this,

$$\begin{aligned} p(X_j)p(X_1)^{m_1}...p(X_k)^{m_k} &= \left(p(X_j)p(X_1)\right)\left(p(X_1)^{(m_1-1)}...p(X_k)^{m_k}\right) \\ &= \left([p(X_j)p(X_1)] + p(X_1)p(X_j)\right)\left(p(X_1)^{(m_1-1)}...p(X_k)^{m_k}\right) \\ &= \left(p([X_jX_1]) + p(X_1)p(X_j)\right)\left(p(X_1)^{(m_1-1)}...p(X_k)^{m_k}\right) \\ &= \left(\sum_{i=1}^{k} L^i_{j1}p(X_i) + p(X_1)p(X_j)\right)\left(p(X_1)^{(m_1-1)}...p(X_k)^{m_k}\right). \end{aligned}$$

Note that without loss of generality it was assumed that $m_1 \geq 1$. Now, realize that the inductive hypothesis applies to the sum

$$\sum_{i=1}^{k} L^i_{j1}p(X_i)p(X_1)^{(m_1-1)}p(X_2)^{m_2}...p(X_k)^{m_k}$$

since, for each i, the term

$$p(X_i)p(X_1)^{(m_1-1)}p(X_2)^{m_2}...p(X_k)^{m_k}$$

is a product of N or less factors. However notice that $\langle K\rangle \leq \langle K+1\rangle$ for each K. Thus

$$\sum_{i=1}^{k} L^i_{j1}p(X_i)p(X_1)^{m_1-1}p(X_2)^{m_2}...p(X_k)^{m_k} \in \langle N+1\rangle.$$

So, from repeated use of the commutator, one can successively 'move' the factor of $p(X_j)$ to the right one position at time while simultaneously generating a new sum that is in $\langle N+1\rangle$. Finally, once the factor of $p(X_j)$ appears in the right position relative to the order of the basis, it will be apparent that $p(X_j)p(X_1)^{m_1}p(X_2)^{m_2}...p(X_k)^{m_k}$ is a linear combination of elements in $\langle N+1\rangle$ with the element

$$p(X_1)^{m_1}p(X_2)^{m_2}...p(X_j)p(X_j)^{m_j}p(X_{j+1})^{m_{j+1}}...p(X_k)^{m_k}.$$

Therefore,

$$p(X_j)p(X_1)^{m_1}p(X_2)^{m_2}...p(X_k)^{m_k}\in\langle N+1\rangle.$$

$\square$

Recall the basis of $\mathfrak{sl}(n,\mathbb{C})$ used in Chapters 3 and 5,

$$\{E_{lk}\mid 1\leq l\neq k\leq n\}\cup\{H_l\mid l\in[n-1]\}.$$

Proposition 6.2.3. *Let $\mathfrak{g}$ equal $\mathfrak{gl}(n,\mathbb{C})$ or $\mathfrak{sl}(n,\mathbb{C})$, and let $p:\mathfrak{g}\to\mathfrak{gl}(W)$ be a complex linear Lie algebra representation for $\mathfrak{g}$. Suppose $w\in W$ is a weight vector for p. Then*

$$\langle w\rangle_p=\langle p(E_{l_1k_1})p(E_{l_2k_2})...p(E_{l_jk_j})w\mid l_i\neq k_i,\ j\in\mathbb{N}\rangle.$$

In particular, the cyclic subspace is spanned by weight vectors.

Proof. To begin. Let $w\in W$ be a weight vector for the representation $p:\mathfrak{sl}(n,\mathbb{C})\to\mathfrak{gl}(W)$, and put some convenient order on the basis

$$\{E_{lk}\mid 1\leq l\neq k\leq n\}\cup\{H_l\mid l\in[n-1]\}$$

such that the elements of $\{H_l\mid l\in[n-1]\}$ appear last in line relative to this order. Temporarily relabel this basis as

$$\mathcal{B}=\{X_1,...,X_{(n^2-1)}\}.$$

So, for each $l\in[n-1]$,

$$X_{n(n-1)+l}=H_l,$$

under the previous requirement.

First, since p is linear, the cyclic subspace of w can be reduced to

$$\langle p(X_{i_1})p(X_{i_2})...p(X_{i_k})w\mid X_{i_j}\in\mathcal{B},\ k\in\mathbb{N}\rangle.$$

Now, let $k\geq 1$, and suppose that the product $p(X_{i_1})p(X_{i_2})...p(X_{i_k})$ contains one or more factors of $p(H_l)$ for $l=1,2...,n-1$. Using Lemma 6.2.2, rewrite the product as

$$p(X_{i_1})p(X_{i_2})...p(X_{i_k})=\sum_{j=1}^{N}a_jp(X_1)^{m_1}p(X_2)^{m_2}...p(X_{n(n-1)})^{m_{n(n-1)}}p(H_1)^{b_1}...p(H_{n-1})^{b_{n-1}}.$$

where all the terms of linear combination satisfy Lemma 6.2.2. However, w is a weight. Thus, for each j, one has that

$$p(H_1)^{b_{1j}}...p(H_{n-1})^{b_{n-1j}}(w)=c_jw,$$

for some appropriate $c_j \in \mathbb{C}$. Hence,

$$p(X_{i_1})p(X_{i_2})...p(X_{i_k})w = \sum_{j=1}^{N} \left(c_j a_j p(X_1)^{m_{1j}} p(X_2)^{m_{2j}} ... p(X_{n(n-1)})^{m_{n(n-1)j}}\right)(w).$$

Therefore, the cyclic subspace of w reduces to

$$\langle w \rangle_p = \langle p(E_{l_1k_1})p(E_{l_2k_2})...p(E_{l_jk_j})w \mid l_i \neq k_i,\ j \in \mathbb{N}\rangle.$$

Furthermore, by Theorem 5.4.8, $p(E_{lk})w$ is another weight vector whenever it is not zero. Therefore, through induction, one sees that $\langle w \rangle_p$ is spanned by weight vectors.

Finally, consider $p : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(W)$. Note that the standard basis of $\mathfrak{gl}(n, \mathbb{C})$ consists of all the matrix units $\{E_{lk} \mid l, k \in [n]\}$. So repeat the last argument with the role of $\mathfrak{h}$ replaced with $\mathfrak{z}$ to get the result. □

Corollary 6.2.4. *Let G be equal to* $\mathrm{SL}(n, \mathbb{C})$ *or* $\mathrm{GL}(n, \mathbb{C})$. *Then an irreducible complex analytic Lie group representation of G is spanned by weight vectors.*

Proof. Suppose $\rho : G \to \mathrm{GL}(W)$ is an irreducible complex analytic representation. Then, by Theorem 4.4.8,

$$\dot{\rho} : \mathfrak{g} \to \mathfrak{gl}(W)$$

is irreducible as well. Now let $w \in W$ be a weight vector, and note that, by Proposition 6.2.3, the cyclic subspace

$$\langle w \rangle_{\dot{\rho}}$$

is spanned by additional weight vectors. However, W is irreducible. Thus,

$$\langle w \rangle_{\dot{\rho}} = W.$$

Therefore, W is spanned by weight vectors. □

This section concludes by showing that the standard basis for $V^{\bigotimes m}$ consists of weight vectors for both $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$, and in addition, that weights and weight spaces for the reducible module $V^{\bigotimes m}$ relative to each $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ are the same. Furthermore, a description of all possible weights using integer composition will be presented.

Let $m = 1$; $j \in [n]$, and let

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in \mathrm{GL}(n, \mathbb{C}).$$

Now

$$D^{\otimes m}(e_j) = d_j e_j.$$

Thus, e_j is a weight vector with weight $\mu_j \equiv (0, ..., 1, ..., 0)$ such that the 1 is in the jth position and there are zeros elsewhere. Furthermore, it is clear that that if D was chosen from $\mathrm{SL}(n, \mathbb{C})$, then one would see the same result. In other words, the matrix Lie groups $\mathrm{GL}(n, \mathbb{C})$, and $\mathrm{SL}(n, \mathbb{C})$ both share the same weight vectors for this case. But as one will see, this naturally applies to every $m \geq 1$.

Building off this example, consider the standard basis for $V^{\bigotimes m}$,

$$\mathcal{E}^m = \{e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \mid I = (i_1, ..., i_m) \in [n]^m\}.$$

Let $I = (i_1, ..., i_m) \in [n]^m$. Then, considering the defining action of the representation

$$T^{\otimes m} : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V^{\bigotimes m}),$$

$e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$ is naturally a weight vector for $\mathrm{GL}(n, \mathbb{C})$. Furthermore, by computing

$$\begin{aligned} D^{\otimes m}(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) &= De_{i_1} \otimes De_{i_2} \otimes ... \otimes De_{i_m} \\ &= d_{i_1}e_{i_1} \otimes d_{i_2}e_{i_2} \otimes ... \otimes d_{i_m}e_{i_m} \\ &= (d_{i_1}d_{i_2}...d_{i_m})e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}. \end{aligned}$$

one can see that the corresponding weight $\mu_I \equiv (m_1, m_2, ..., m_n)$ is defined by

$$m_i = \text{'the number of factors of } e_i\text{.'}$$

As a consequence, $m_1 + m_2 + ... + m_n = m$, holds for any weight μ_I. Moreover, since $\mu_I \equiv (m_1, m_2, ..., m_n)$ consists of an *order sequence of nonnegative integers*, it follows that all the distinct weights of $T^{\otimes m}$ are determined by all possible *integer compositions of m*. These will be denoted as $\mathbf{m}$ from this point on.

Now, if $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$ and $e_{j_1} \otimes e_{j_2} \otimes ... \otimes e_{j_m}$ are in the same weight space, then the fact that $\mu_I = \mu_J$ implies $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$ and $e_{j_1} \otimes e_{j_2} \otimes ... \otimes e_{j_m}$ share the same number of factors of each e_i. Thus,

$$e_{j_1} \otimes e_{j_2} \otimes ... \otimes e_{j_m} = \sigma e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$$

for some permutation $\sigma \in \mathcal{S}_m$. Therefore, if $V_{\mathbf{m}}$ denotes the weight space associated with $\mu_{\mathbf{m}}$, then

$$V_{\mathbf{m}} = \langle \sigma e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \mid \sigma \in \mathcal{S}_m \rangle,$$

such that $1 \leq i_1 \leq i_2 ... \leq i_m \leq n$.

Finally, each $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$ is also a weight vector for $\mathrm{SL}(n, \mathbb{C})$. In addition, two distinct weights $\mu_{\mathbf{m_1}}$ and $\mu_{\mathbf{m_2}}$ for $\mathrm{GL}(n, \mathbb{C})$ will restrict down to two distinct weights of $\mathrm{SL}(n, \mathbb{C})$. Indeed, if

$$\mu_{\mathbf{m_1}} \equiv (m_1, ..., m_n) \quad \text{and} \quad \mu_{\mathbf{m_2}} \equiv (\overline{m}_1, ..., \overline{m}_n)$$

are to restrict down to the same weight for $\mathrm{SL}(n, \mathbb{C})$, then there must exist an integer a, such that

$$(m_1 - \overline{m}_1, m_2 - \overline{m}_2, ..., m_n - \overline{m}_n) = (a, a, ..., a).$$

However, the condition that

$$m_1 + ... + m_n = \overline{m}_1 + ... + \overline{m}_n = m$$

implies

$$an = (m_1 - \overline{m}_1) + (m_2 - \overline{m}_2) + ... + (m_n - \overline{m}_n) = 0.$$

Thus, it must be that $a = 0$. Therefore, $\mu_{\mathbf{m_1}} = \mu_{\mathbf{m_2}}$ as weights for $\mathrm{GL}(n, \mathbb{C})$ since $m_i = \overline{m}_i$ for each i.

There is subtle point to made from this. Suppose that $v \in V^{\bigotimes m}$ was only assumed to be a weight vector for $\mathrm{SL}(n, \mathbb{C})$ with its corresponding weight being ν Then first, ν is determined by some family

$$\{(m_1, m_2..., m_n) + (a, a, ..., a) \mid a \in \mathbb{Z}\}$$

for some $(m_1, m_2..., m_n)$. Now by Lemma 5.1.3, weight vectors from different weight spaces are linearly independent. However, the weight spaces for $\mathrm{GL}(n, \mathbb{C})$ are all given by

$$V_{\mathbf{m}} = \langle \sigma e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \mid \sigma \in \mathcal{S}_m \rangle$$

as $\mathbf{m}$ ranges over all the compositions of m. So, if ν was distinct from various restrictions of $V_{\mathbf{m}}$ to $\mathrm{SL}(n, \mathbb{C})$, then a contradiction would arise: v would be linearly independent from the space

$$\bigoplus_{\mathbf{m}} V_{\mathbf{m}} = V^{\bigotimes m}.$$

Finally, it was just shown that only one distinct $\mu_{\mathbf{m}}$ could restrict down to ν. Therefore, for some $\mathbf{m}$, one has that v is a weight vector for $\mathrm{GL}(n, \mathbb{C})$ with weight $\mu_{\mathbf{m}} \equiv (m_1, ..., m_n) = \mathbf{m}$.

6.3 The image space $\epsilon_t \left(V^{\bigotimes m}\right)$ as an irreducible $\mathrm{SL}(n, \mathbb{C})$-module

This section starts by finishing the verification that, for the tableau t corresponding to the partition $\lambda \vdash m$, the corresponding $\mathrm{GL}(n, \mathbb{C})$ and $SL(N, \mathbb{C})$-module,

$$\epsilon_t \left(V^{\bigotimes m}\right),$$

possess the same weights and weight spaces as seen from each setting of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$.

Now, like any linear transformation, one can appeal to the span of the image set of the basis $\mathcal{E}^m$. This is seen as

$$\epsilon_t \left(V^{\bigotimes m}\right) = \langle \epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m}) \mid I = (i_1, ..., i_m) \in [n]^m \rangle.$$

Now by Corollary 6.2.4, as an irreducible $\mathrm{GL}(n, \mathbb{C})$-module, $\epsilon_t \left(V^{\bigotimes m}\right)$ has a basis of weight vectors. But, since the action of $\mathrm{GL}(n, \mathbb{C})$ commutes with the action of $\mathcal{S}_m$ on $V^{\bigotimes m}$, one needs to look no further than the set

$$\{\epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m}) \mid I = (i_1, ..., i_m) \in [n]^m\}.$$

True, take any

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in \mathrm{GL}(n, \mathbb{C}).$$

Then,

$$D^{\otimes m}\left(\epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m})\right) = \epsilon_t\left(D^{\otimes m}(e_{i_1} \otimes ... \otimes e_{i_m})\right) = \mu_I(D)\epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m}).$$

Hence, the weight vectors for $\epsilon_t\left(V^{\bigotimes m}\right)$ are by given all the elements $\epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m})$ such that $\epsilon_t(e_{i_1} \otimes ... \otimes e_{i_m}) \neq 0$. Consequently, the distinct weight spaces in $\epsilon_t\left(V^{\bigotimes m}\right)$ are given by the compositions $\mathbf{m} = (m_1, ..., m_n)$ such that

$$\epsilon_t(V_{\mathbf{m}}) \neq \{0\}.$$

Finally, note that the weight vectors of $\mathrm{SL}(n, \mathbb{C})$ in $\epsilon_t\left(V^{\bigotimes m}\right)$ must also be weight vectors of $\mathrm{GL}(n, \mathbb{C})$ since this result has been established for all of $V^{\bigotimes m}$.

So with this observation, comes the first of the major theorems of this chapter.

Theorem 6.3.1. *Let $\lambda \vdash n$, and suppose $\epsilon_t\left(V^{\bigotimes m}\right) \neq 0$. Then, $\epsilon_t\left(V^{\bigotimes m}\right)$ carrying the representation $T^{\otimes m}$ restricted to* $\mathrm{SL}(n, \mathbb{C})$ *is an irreducible* $\mathrm{SL}(n, \mathbb{C})$*-module.*

Proof. Suppose $\epsilon_t\left(V^{\bigotimes m}\right)$ is nontrivial. Then, by Lemma 5.3.1, find $W \leq \epsilon_t\left(V^{\bigotimes m}\right)$, a nonzero irreducible $\mathrm{SL}(n, \mathbb{C})$-submodule. By Corollary 6.2.4, W has a basis of weight vectors for $\mathrm{SL}(n, \mathbb{C})$. So, denote this basis as $\{w_1, ...w_k\}$, with $k = \dim(W)$, and let $A \in \mathrm{GL}(n, \mathbb{C})$. Using A, define

$$D_A := \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

and $\hat{A} := AD_{A^{-1}}$. By construction, $A = \hat{A}D_A$ where $\hat{A} \in \mathrm{SL}(n, \mathbb{C})$. Indeed,

$$(D_A)^{-1} = D_{A^{-1}},$$

and

$$\det(\hat{A}) = \det(A)\det(D_{A^{-1}}) = \det(A)\det(A^{-1}) = 1.$$

With this in mind, let w_i be the ith basis element of W, and suppose μ_i is its weight. Now, it has been established that weight vectors for $\mathrm{SL}(n, \mathbb{C})$ are also weight vectors of $\mathrm{GL}(n, \mathbb{C})$, and that there exists $(m_1, ..., m_n)$ as a representative for μ_i given by a weight for $\mathrm{GL}(n, \mathbb{C})$.

Thus,

$$\begin{aligned}
A^{\otimes m}(w_i) &= T^{\otimes m}(\hat{A}D_A)(w_i) \\
&= \hat{A}^{\otimes m}\left(T^{\otimes m}(D_A)(w_i)\right) \\
&= \hat{A}^{\otimes m}(\det(A)^{m_1}w_i) \\
&= \det(A)^{m_1}\left(\hat{A}^{\otimes m}w_i\right).
\end{aligned}$$

But

$$\hat{A}^{\otimes m}w_i \in W$$

since W is a $\mathrm{SL}(n,\mathbb{C})$-submodule, and $\hat{A} \in \mathrm{SL}(n,\mathbb{C})$. Consequently,

$$A^{\otimes m}(w_i) = \det(A)^{m_1}\left(\hat{A}^{\otimes m}w_i\right) \in W.$$

From this, one can see that W is also a nonzero $\mathrm{GL}(n,\mathbb{C})-$submodule, and therefore, W must equal $\epsilon_t\left(V^{\bigotimes m}\right)$. □

From this comes the obvious, the need of necessary and sufficient conditions in determining whether or not a particular image space of a general projection operator is nontrivial. Fortunately, this is next in line.

Theorem 6.3.2. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$, and let t be a tableau corresponding to λ. Then, $\epsilon_t\left(V^{\bigotimes m}\right) \neq 0$ if and only if $\dim V = n \geq l$.*

Proof. Let t be a tableau of shape λ. Recall that

$$R_i = \{t_{i,j} \mid j \in [\lambda_i]\} \text{ and } C_j = \{t_{i,j} \mid i \in [\lambda_j^*]\}$$

where $\lambda_j^* = \max\{k \in [l] \mid \lambda_k \geq j\}$ for each $j \in [\lambda_1]$. Now if $n \geq l$, then it is easy to find a nonzero element in $\epsilon_t\left(V^{\bigotimes m}\right)$. Indeed, by this assumption, one can take the first l basis vectors $\{e_1, e_2, ..., e_l\}$, and pair them to each of the rows $\{R_1, R_2, ..., R_l\}$, *respectively*. In other words, e_1 pairs with R_1, e_2 pairs with R_2, and etc. Note how this would not be possible if $n < l$. With this mind, find the monomial, $\mathbf{e}^t := e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$, such that, for each $i \in [l]$, the factors of e_i exist in all the positions given by the entries in R_i. For example, if $n = 3$, $m = 6$ and

$$t \equiv \begin{array}{l} 436 \\ 25 \\ 1 \end{array}$$

then,

$$\mathbf{e}^t = e_3 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1$$

would be the monomial of interest.

By construction, one has that $\sigma\mathbf{e}^t = \mathbf{e}^t$, for all $\sigma \in R_t$ and hence

$$\iota_t(\mathbf{e}^t) = \sum_{\sigma \in R_t} \sigma\mathbf{e}^t = |R_t|\mathbf{e}^t.$$

However, also by construction, for each i, no two factors are the same in any of the positions given by the entries of C_i. Thus, $\sigma\mathbf{e}^t \neq \mathbf{e}^t$ for each $\sigma \in C_t \setminus \{\varepsilon\}$, and, for all $\sigma, \tau \in C_t$, it is true that $\sigma\mathbf{e}^t = \tau\mathbf{e}^t$ if and only if $\sigma = \tau$. As a result,

$$\kappa_t(\mathbf{e}^t) = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma\mathbf{e}^t \neq 0$$

since $\{\sigma\mathbf{e}^t \mid \sigma \in C_t\}$ is a linear independent set. Therefore,

$$\epsilon_t(\mathbf{e}^t) = \kappa_t\iota_t(\mathbf{e}^t) = |R_t|\kappa_t(\mathbf{e}^t) \neq 0.$$

Conversely, suppose that $n < l$, and let $I = (i_1, i_2, ...i_m) \in [n]^m$. Then, by a standard pigeonhole argument, there exists a e_j and a pair $k_1 \neq k_2 \in C_1 = \{t_{i,1} \mid i \in [l]\}$, such that

$$e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}$$

has a factor of e_j in the k_1th and k_2th position. Without loss, assume $k_1 < k_2$, and let $\sigma = (k_1, k_2)$. Then, $\sigma \in C_t$, and

$$\begin{aligned} \sigma(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) &= \sigma(e_{i_1} \otimes ... \otimes e_{k_1} \otimes ... \otimes e_{k_2} \otimes ... \otimes e_{i_m}) \\ &= e_{i_1} \otimes ... \otimes e_{k_2} \otimes ... \otimes e_{k_1} \otimes ... \otimes e_{i_m} \\ &= e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \end{aligned}$$

since $e_{k_1} = e_{k_2} = e_j$. With this in mind, notice that

$$\kappa_t\sigma = \sum_{\tau \in C_t} \mathrm{sgn}(\tau)\tau\sigma = \mathrm{sgn}(\sigma)\left(\sum_{\tau \in C_t} \mathrm{sgn}(\tau\sigma)\tau\sigma\right) = -\kappa_t$$

since

(1) $\mathrm{sgn}(\sigma)^2 = 1$,

(2) $\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma) = \mathrm{sgn}(\tau\sigma)$, and

(3) $\sum_{\tau \in C_t} \mathrm{sgn}(\tau\sigma)\tau\sigma = \sum_{\tau \in C_t} \mathrm{sgn}(\tau)\tau$.

Thus,

$$\begin{aligned} \kappa_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) &= \kappa_t(\sigma e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \\ &= (\kappa_t\sigma)(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \\ &= -\kappa_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}). \end{aligned}$$

However, if this is the case, then one would have

$$2\kappa_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) = 0.$$

Therefore,

$$\kappa_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) = 0.$$

Now, let $\pi \in R_t$. Then, by repeating the previous pigeonhole argument for the monomial

$$e_{i_{\pi^{-1}(1)}} \otimes e_{i_{\pi^{-1}(2)}} \otimes ... \otimes e_{i_{\pi^{-1}(m)}} = \pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}),$$

one would also find

$$\kappa_t(e_{i_{\pi^{-1}(1)}} \otimes e_{i_{\pi^{-1}(2)}} \otimes ... \otimes e_{i_{\pi^{-1}(m)}}) = 0.$$

Thus,

$$\begin{aligned}
\epsilon_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) &= \kappa_t \iota_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \\
&= \kappa_t \left(\sum_{\pi \in R_t} \pi e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \right) \\
&= \left(\sum_{\pi \in R_t} \kappa_t(\pi e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \right) \\
&= 0.
\end{aligned}$$

Therefore,

$$\epsilon_t \left(V^{\bigotimes m} \right) = \{0\}.$$

□

Corollary 6.3.3. *Let* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$*. Then* $U^\lambda \neq 0$ *if and only if* $\dim V = n \geq l$*.*

Proof. This follows by Theorem 6.3.2 and Theorem 3.4.2. □

Now, the following result will needed for the remaining objective of identifying highest weight vectors and their corresponding highest weights in each of the nontrivial image spaces of the general projection operators corresponding to the various integer partitions of m.

Lemma 6.3.4. *Let* t *be a tableau of shape* $\lambda \vdash m$*, and let* $I = (i_1, i_2, ..., i_m) \in [n]^m$*. Suppose that, for all* $\pi \in R_t$*, there exists a column* C_j *of* t *and a basis vector* $e_k \in V$*, such that*

$$\pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) = e_{i_{\pi^{-1}(1)}} \otimes e_{i_{\pi^{-1}(2)}} \otimes ... \otimes e_{i_{\pi^{-1}(m)}}$$

has a factor of e_k *in at least two of the positions given by the entries of* C_j*. Then,*

$$\epsilon_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) = 0.$$

Proof. Let $I = (i_1, i_2, ...i_m) \in [n]^m$, and $\pi \in R_t$. Find such a column C_j of t, and a basis vector $e_k \in V$, such that, for some pair $k_1 < k_2 \in C_j$,

$$\pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) = e_{i_{\pi^{-1}(1)}} \otimes e_{i_{\pi^{-1}(2)}} \otimes ... \otimes e_{i_{\pi^{-1}(m)}}$$

has a factor of e_k in both the k_1th and k_2th positions. Then, $\sigma = (k_1, k_2) \in C_t$, and

$$\sigma\left(\pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m})\right) = \pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}).$$

But, by considering the proof of the Theorem 6.3.2, one has

$$\kappa_t(\pi(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m})) = 0.$$

Therefore, it follows that

$$\begin{aligned}
\epsilon_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) &= \kappa_t \iota_t(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \\
&= \kappa_t \left(\sum_{\pi \in R_t} \pi e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m} \right) \\
&= \left(\sum_{\pi \in R_t} \kappa_t(\pi e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_m}) \right) \\
&= 0.
\end{aligned}$$

□

This section finally arrives at the last major result of the chapter preceding the formal presentation of the classifications theorems regarding the specific irreducible representations corresponding to $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SU}(n)$. So, recall that t_λ denotes the standard tableau of shape λ, and that

$$R_{t_\lambda} = \mathcal{S}_{\{1,2,..,\lambda_1\}} \times \mathcal{S}_{\{\lambda_1+1,\lambda_2+2...,\lambda_1+\lambda_2\}} \times ... \times \mathcal{S}_{\{n-\lambda_{l-1}+1, n-\lambda_{l-1}+2,...n\}}.$$

Theorem 6.3.5. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$ such that $n \geq l$, and let t be a tableau corresponding to λ. Then, the highest weight in $\epsilon_t\left(V^{\bigotimes m}\right)$ is given by*

$$\mathbf{m}_\lambda = (\lambda_1, \lambda_2, ..., \lambda_l, 0, ..., 0).$$

Proof. The proof will be established via $\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$. This will suffice, since each $\epsilon_t\left(V^{\bigotimes m}\right)$ is isomorphic to one another.

By Theorem 5.3.3, a weight vector v is a highest weight vector if and only if v is left fixed by $\mathcal{U}$, the subgroup of upper triangular unipotent matrices. Also, by Theorem 6.3.2,

$$\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right) \neq \{0\}$$

since $n \geq l$. Thus, there is a unique one dimensional weight space of highest weight. So, the proof amounts to finding a highest weight vector.

Now, considering the proof of Theorem 6.3.2, the element,

$$\mathbf{e}_\lambda := \underbrace{e_1 \otimes ... \otimes e_1}_{\lambda_1} \otimes \underbrace{e_2 \otimes ... \otimes e_2}_{\lambda_2} \otimes ... \otimes \underbrace{e_{l-1} \otimes ... \otimes e_{l-1}}_{\lambda_{l-1}} \otimes \underbrace{e_l \otimes ... \otimes e_l}_{\lambda_l},$$

satisfies $\epsilon_t(\mathbf{e}_\lambda) \neq 0$. Now, denote its weight as $\mu_{\mathbf{m}_\lambda}$, and see that

$$\mu_{\mathbf{m}_\lambda} \equiv (\lambda_1, \lambda_2, ..., \lambda_l, 0, ..., 0).$$

Therefore, one just needs to show that $U^{\otimes m}(\epsilon_t(\mathbf{e}_\lambda)) = \epsilon_t(\mathbf{e}_\lambda)$ for all

$$U = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{U}.$$

First, $\mathcal{U}$ is generated by $\{I + aE_{ij} \mid a \in \mathbb{C}, 1 \leq i < j \leq n\}$. Thus, one can reduce the problem to just considering this generating set. Furthermore, the largest ordered basis appearing in $\mathbf{e}_\lambda$ is e_l where $l \leq n$. So, again the problem is reduced to a smaller setting, this being

$$\{I + aE_{ij} \mid a \in \mathbb{C}, 1 \leq i < j \leq l\}.$$

Second, it will be convenient to define

$$a^K := a^{|\{k_N | k_N = i, N \in [\lambda_j]\}|}$$

for each

$$K = (k_1, k_2, ..., k_{\lambda_j}) \in \{i, j\}^{\lambda_j},$$

and

$$\mathbf{e}_{(j \to i)} := \sum_{K \in \{i,j\}^{\lambda_j} \setminus \{(j,...,j)\}} \left(a^K \underbrace{e_1 \otimes ... \otimes e_1}_{\lambda_1} \otimes ... \otimes e_{k_1} \otimes e_{k_2} \otimes ... \otimes e_{k_{\lambda_j}} \otimes ... \otimes \underbrace{e_l \otimes ... \otimes e_l}_{\lambda_l} \right).$$

With this in mind, realize that

$$(I + aE_{ij})^{\otimes m} (\mathbf{e}_\lambda) = \mathbf{e}_\lambda + \mathbf{e}_{(j \to i)}.$$

Thus, one has

$$(I + aE_{ij})^{\otimes m} (\epsilon_t(\mathbf{e}_\lambda)) = \epsilon_t \left((I + aE_{ij})^{\otimes m} (\mathbf{e}_\lambda) \right) = \epsilon_t(\mathbf{e}_\lambda) + \epsilon_t(\mathbf{e}_{(j \to i)}).$$

Now, if

$$K = (k_1, k_2, ..., k_{\lambda_j}) = \{i, j\}^{\lambda_j} \setminus \{(j, ..., j)\},$$

then, for some $N \in [\lambda_j]$, it is true that $k_N = i$. Thus, the monomial

$$\underbrace{e_1 \otimes ... \otimes e_1}_{\lambda_1} \otimes \underbrace{e_2 \otimes ... \otimes e_2}_{\lambda_2} \otimes ... \otimes e_{k_1} \otimes ... \otimes e_{k_N} \otimes ... \otimes e_{k_{\lambda_j}} \otimes ... \otimes \underbrace{e_l \otimes ... \otimes e_l}_{\lambda_l}$$

will have a factor of e_i in at least two of the positions given by the entries of the column C_N. Indeed $i < j$. So, by construction of the original $\mathbf{e}_\lambda$, there will be a factor of e_i in the

$(t_\lambda)_{iN}$th position, along with the factor of e_i in the $(t_\lambda)_{iN}$th position. $((t_\lambda)_{iN}, (t_\lambda)_{jN} \in C_N)$ Furthermore, since there are only factors of e_i in the positions $\lambda_{i-1} + 1$ through $\lambda_{i-1} + \lambda_i$, and since $\lambda_i \geq \lambda_j$, one will find that, for each $\pi \in R_t$, there will also be a factor of e_i in at least two of the positions given by the entries of the column $C_{\pi^{-1}(N)}$. Consequently, by Lemma 6.3.4,

$$\epsilon_{t_\lambda}\left(\underbrace{e_1 \otimes ... \otimes e_1}_{\lambda_1} \otimes \underbrace{e_2 \otimes ... \otimes e_2}_{\lambda_2} \otimes ... \otimes e_{k_1} \otimes ... \otimes e_{k_N} \otimes ... \otimes e_{k_{\lambda_j}} \otimes ... \otimes \underbrace{e_l \otimes ... \otimes e_l}_{\lambda_l}\right)$$

is equal to zero. As a result, $\epsilon_t(\mathbf{e}_{(j\to i)}) = 0$, and therefore,

$$(I + aE_{ij})^{\otimes m}\left(\epsilon_t(\mathbf{e}_\lambda)\right) = \epsilon_t(\mathbf{e}_\lambda) + \epsilon_t(\mathbf{e}_{(j\to i)}) = \epsilon_t(\mathbf{e}_\lambda).$$

□

With the establishment of Theorem 6.3.5, the work of this exposition is complete. What remains is pooling together all the major results established here along with major theorems from Chapters 3, 4, and 5. This begins with the complete classification of irreducible complex analytic representations for $\mathrm{SL}(n, \mathbb{C})$.

Theorem 6.3.6. *For all $n \geq 1$, each finite dimensional complex analytic irreducible representations of $\mathrm{SL}(n, \mathbb{C})$ can be realized in*

$$\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$$

for some positive integer m and partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$ satisfying $n - 1 \geq l$.

Furthermore, the collection

$$\left\{\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right) \mid \lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m,\ \ n-1 \geq l,\ \ m \geq 1\right\}$$

forms a complete list of irreducible complex analytic representations of $\mathrm{SL}(n, \mathbb{C})$.

Proof. Let $n \geq 1$, and first suppose $\rho : \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(W)$ is a complex analytic irreducible representation carried by the finite dimensional vector space, W. Then, by Theorem 5.3.8, ρ is uniquely identified by its highest weight, μ. Furthermore, applying Theorem 5.4.13 to μ, one sees that ρ is determined uniquely by some sequence of integers,

$$(m_1, m_2, ..., m_{n-1}, 0)$$

satisfying $m_1 \geq m_2 \geq, ..., \geq m_{n-1} \geq 0$. However, if

$$m := m_1 + m_2 + ... + m_{n-1},$$

and

$$\lambda := (m_1, m_2, ..., m_{n-1}, 0),$$

then, by Theorem 6.3.5, W is equivalent to the $\mathrm{SL}(n, \mathbb{C})$-module, $\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$, carrying the representation $T^{\otimes m}$ restricted to $\mathrm{SL}(n, \mathbb{C})$. Therefore, ρ is realized in $\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$ for $\lambda \vdash m$ defined above.

Now let $m \geq 1$, and suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ is an integer partition of m satisfying $n-1 \geq l$. Then, by Theorem 6.3.2, $\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right) \neq 0$. Thus, using Theorem 6.3.1 and 6.3.5, one has that, as an $\mathrm{SL}(n, \mathbb{C})$-module, $\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$, carrying the representation $T^{\otimes m}$ restricted to $\mathrm{SL}(n, \mathbb{C})$, is irreducible, and has a highest weight given by $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$. Finally, considering Theorem 5.3.8, one sees that λ and m uniquely determine a complex analytic irreducible representation for $\mathrm{SL}(n, \mathbb{C})$. $\square$

6.4 Highest weight classification for $\mathrm{SU}(n)$

Before the exposition concludes with the main objective, being the classification theorem for the irreducible representations of $\mathrm{SU}(n)$, a small issue concerning the potential definition of weight and weight space for the setting of $\mathrm{SU}(n)$ needs to be resolved. At first glance, the concept of highest weight and highest weight space doesn't seem to apply to Lie group representations of $\mathrm{SU}(n)$ since the matrix Lie groups $\mathcal{U}$ and $\mathcal{L}$ are not subgroups of $\mathrm{SU}(n)$. On the other hand, Lie's Theorem does apply to

$$\mathcal{D} \cap \mathrm{SU}(n),$$

which consists of matrices of form

$$D = \begin{bmatrix} \mathrm{e}^{i\theta_1} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{i\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{e}^{i\theta_n} \end{bmatrix}$$

such that $\theta_1 + \theta_2 + ... + \theta_n \equiv 0 \pmod{2\pi}$. Thus, weights and weight spaces can have meaning for $\mathrm{SU}(n)$. With this, and with Theorems 4.4.14 and 4.4.15, one ultimately may extend the notion of highest weight and highest weight space to the setting of $\mathrm{SU}(n)$ with the aid of $\mathrm{SL}(n, \mathbb{C})$. Therefore, one has the following, long anticipated classification theorem for irreducible representations of $\mathrm{SU}(n)$.

Theorem 6.4.1. *For all $n \geq 1$, each finite-dimensional irreducible representation of $\mathrm{SU}(n)$ can be realized in*

$$\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$$

for some positive integer m and partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$ satisfying $n - 1 \geq l$.

Furthermore, the collection

$$\left\{\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right) \mid \lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m, \;\; n-1 \geq l, \;\; m \geq 1\right\}$$

forms a complete list of irreducible representations of $\mathrm{SU}(n)$.

Proof. Let $n \geq 1$, and first suppose $\rho : \mathrm{SU}(n) \to \mathrm{GL}(W)$ is a irreducible representation carried by the finite dimensional vector space, W. Then, by Theorem 4.4.15 and 6.3.6, ρ can be realized in

$$\epsilon_{t_\lambda}\left(V^{\bigotimes m}\right)$$

for some positive integer m and partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \vdash m$ satisfying $n - 1 \geq l$.

Now let $m \geq 1$, and suppose $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$ is an integer partition of m satisfying $n - 1 \geq l$. Then, by Theorem 6.3.6 and 4.4.14, there exists an irreducible representation of $\mathrm{SU}(n)$ determined uniquely by λ and m. $\square$

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