MAT 320
12/9/2017

Review topics for Final Exam

I. Formal logic

A. propositional logic

i. propositions and connectives; $\vee, \wedge, \sim, \implies, \iff$

ii. tautologies (valid arguments)

(a) DeMorgan's Laws

(b) standard arguments, e.g., *modus ponens*, contrapositive (or indirect) proof, proof by contradiction.

B. predicate logic

i. open sentences (or predicates); $\forall$ and $\exists$; interpretation (English to symbols, symbols to English); free and bound variables; the difference between $(\forall x)(\exists y)P(x, y)$ and $(\exists y)(\forall x)P(x, y)$.

ii. inference rules *

C. DeMorgan's Laws for quantified predicates.

II. Set theory

A. definition of subset and equality of sets, the difference between $\in$ vs. $\subseteq$

B. empty set, intersection, union, complement, difference; DeMorgan's Laws for pairs of sets; set-theoretic identities and elementary theorems

C. new sets from old

i. power sets

ii. cartesian product of sets

D. families of sets

i. families and indexed families

ii. intersection and union

iii. DeMorgan's Laws for families

E. Axiomatic set theory

i. Russell's paradox and the separation axiom *

ii. the axiom of infinity and the construction of natural numbers

iii. The principle of mathematical induction

iv. the axiom of choice *

F. Mathematical induction

i. inductive sets*, well-ordering property of the natural numbers

ii. "ordinary" (one-step) induction

iii. "strong" induction

III. Relations on a set

A. definition of relation on X as set of ordered pairs; representation as directed graph

B. equivalence relations

i. reflexivity, symmetry, transitivity

ii. equivalence classes and the partition associated with an equivalence relation

* excluded from final exam

iii. relation associated with a given partition

iv. well-definition of functions and operations defined in terms of representatives of equivalence classes

C. Number systems

i. construction of integers and rational numbers as sets of equivalence classes [*]

ii. "congruence modulo n" and $\mathbb{Z}_n$; modular addition and multiplication

D. order relations

i. antisymmetric relations and partially ordered sets

ii. linear (total) orderings, upper and lower bounds, lub's and glb's, maximal and minimal elements.

iii. Hasse diagrams

iv. standard examples: the Boolean lattice $\mathcal{P}(X)$ (ordered by inclusion), $\mathbb{N}$ ordered by divisibility; the partition lattice

IV. Relations between sets

A. definition of "relation from X to Y"

B. source and domain, codomain (target) and range

C. inverse relations

D. identity relation

E. composition of relations

(i) relations as "partially-defined, multi-valued functions"; definition of composition of relations

(ii) expression of reflexive, symmetric, transitive, and anti-symmetric properties of a relation R in terms of composition, inverse, and identity relations

(iii) associativity of composition

(iv) composites of inverse relations, esp. when one is a function, and relation with injectivity and surjectivity

V. Functions as relations

A. definition; functional notation

i. domain and codomain

ii. vertical line test; single-valued relations

B. standard constructions

i. identity function

ii. the inclusion map of a subset to a set; restriction of a function to a subset of the domain; "co-restriction" of the codomain to a subset containing the range

iii. the canonical projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$

iv. the canonical quotient map $q\colon X \to X/\sim$ arising from an equivalence relation $\sim$; equivalently, the natural map $\pi\colon X \to \mathcal{B}$ associated to a partition $\mathcal{B}$ of X

C. injectivity and surjectivity

i. definitions; "horizontal line test" for injectivity; "vertical line test (variant)" for surjectivity

ii. the four theorems describing behavior under compositions (injectivity/surjectivity of f and/or g implies/is implied by injectivity/surjectivity of $f \circ g$)

[*] excluded from final exam

D. bijections and invertible functions

i. inverse relations and inverse functions

ii. definition of invertible function; equivalence of invertibility and bijectivity

iii. permutations of a set; two-line, one-line, and cycle notation; composition and inversion

iv. the symmetric group, and abstract groups*

E. induced set functions

i. the image $f(A)$ (written $f_\vdash(A)$ in our class) of a subset A of the domain of f

ii. the pre-image (*a.k.a* inverse image) $f^{-1}(B)$ (written $f^\dashv(A)$ in our class) of a subset B of the codomain of f

iii. behavior under composition: $(f \circ g)_\vdash = f_\vdash \circ g_\vdash$ and $(f \circ g)^\dashv = g^\dashv \circ f^\dashv$

iv. behavior of $f_\vdash$ and $f^\dashv$ with respect to union, intersection, complement, difference (of pairs or families of sets) for general functions, injective functions, and surjective functions

v. order-preserving functions between partially-ordered sets

vi. continuous functions between topological spaces

F. functions, partitions, and equivalence relations

(i) the partition of the domain given by the fibers of a function

(ii) the equivalence relation on the domain determined by a function

(iii) well-definition and bijectivity of the induced map: the "first isomorphism theorem for set functions"*

G. finite topological spaces and posets

(i) posets

(a) isomorphisms of partially-ordered sets (posets)

(b) order-preserving maps of posets

(c) upper ideals in posets

(ii) finite topological spaces

(a) definition of (finite) topological space and continuous function

(b) topologies on small finite sets

(c) the order topology (of upper ideals) on a poset

(d) continuity of order-preserving maps

VI. Cardinality

A. finite and infinite sets

(i) definition of equicardinal sets using bijections

(ii) finite sets

(a) pigeonhole principle and non-existence of bijection $f\colon \{1,\ldots,n\} \to \{1,\ldots,m\}$ for $n \neq m$

(b) injectivity/surjectivity of $f\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ implies the other

(c) definition of finite set and cardinality of a finite set

(iii.) equicardinality of $(-\infty,\infty)$ and (a,b)

B. countable and uncountable sets

i. definition of countability

ii. countability of $\mathbb{N} \times \mathbb{N}$

iii. countability of $\mathbb{Z}$ and $\mathbb{Q}$

iv. countable unions of countable sets

*excluded from final exam

v. nonexistence of bijection $X \to \mathcal{P}(X)$

vi. uncountability of $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$: Cantor's diagonal argument

vii. equicardinality of $(0,1)$ and $\mathcal{P}(\mathbb{N})$: characteristic functions, bit strings, and binary decimal expansion*

vii. "cardinal order" ($|A| \leq |B|$); axiom of choice and Schröder-Bernstein Theorem *

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